Kvadratna metoda M_{split} v deformacijski analizi na primeru 2D geodetske mreže

Squared M_{split} Estimation in Deformation Analysis – 2D Geodetic Network Case Study

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IZVLEČEK

Deformacijska analiza je kompleksen postopek, kjer na podlagi vsaj dveh periodičnih izmer odkrivamo in določamo prostorske premike točk v obravnavani geodetski mreži. S tem ugotavljamo premike in deformacije grajenega in naravnega okolja. V članku je obravnavana kvadratna metoda M_{split}, ki je nadgradnja metode največjega verjetja. Izpeljane so enačbe kvadratne metode M_{split}. Podan je prikaz metode na testnem primeru 2D geodetske mreže po izpeljanih enačbah, dodatno je na istem testnem primeru izvedena še primerjalna analiza z rezultati drugih postopkov deformacijske analize. Rezultati kvadratne metode M_{split} se nekoliko razlikujejo od simuliranih, največja razlika na točkah, ki so se premaknile, je 11,5 mm, na točkah, ki so pri miru, pa 10,4 mm, kar so zadovoljivi rezultati. Ugotavljamo, da kvadratna metoda M_{split} vrne rezultate, primerljive drugim metodam, zato ocenjujemo, da je uporabna za deformacijsko analizo in je lahko eden od postopkov deformacijske analize.

ABSTRACT

Deformation analysis is a complex procedure where, based on several periodic geodetic measurements, displacements of points in the geodetic network are detected and determined. On this basis, movements and deformations of the built and natural environment are detected. The article discusses the Squared M_{split} estimation, an extension of the maximum likelihood method, which is one of the procedures used in deformation analysis. The equations of the Squared M_{split} estimation are derived and the method is presented on 2D geodetic network case study. The effectiveness of the presented method is compared with the results of other deformation analysis approaches performed with the same numerical example. The results obtained using the Squared M_{split} estimation slightly differ from the simulated values, with the maximum discrepancy being 11.5 mm at unstable points and 10.4 mm at stable points, which are satisfactory results. The findings indicate that the Squared M_{split} estimation provides results comparable to other methods. Therefore, it is considered suitable for deformation analysis and can be regarded as one of the applicable procedures in this field.

KLJUČNE BESEDE

deformacijska analiza, geodetska izmera, kvadratna metoda M_{split}, iterativni postopek, premiki točk geodetske mreže

KEY WORDS

deformation analysis, geodetic surveys, Squared M_{split} estimation, iterative process, point displacement in the geodetic network

1 INTRODUCTION

Deformation analysis is based on the periodic measurement of geodetic monitoring networks, to detect and determine changes in the spatial position of points connected to geodetic networks, and thus the geometrical changes of built structures (hydroelectric power plants, chimneys, bridges, etc.), the environment and natural objects (landslides, rock/soil landfills, etc.). The reference points determine the geodetic datum of the network, which is theoretically defined as the smallest number of given quantities needed to determine the coordinates of the geodetic point in the selected coordinate system. They are used to define the position, orientation and scale of the geodetic network (Pleterski, 2022). Problems in the interpretation of the results may arise in case of incorrect assumptions about the stability of reference points of the geodetic network. In practical applications, we aim to position reference points on a stable region, outside the influence zone of the deformable object under study, in order to ensure their stability and immobility throughout the duration of the analysis. Equally important is the appropriate geometrical configuration, which enables the optimal distribution of errors in the geodetic network. Control points are usually permanently stabilized on the studied object. Their locations are determined in collaboration with experts from other fields. Based on the determined movement trajectories of the control points, it is possible to conclude what is happening with the studied object and to warn of potential dangers.

In geodetic practice, deformation analysis methods are often considered too difficult due to their complexity and mathematical background, so the method of determining the displacement of object points is simplified. Therefore, the test is often used to determine the statistical characteristic of the displacement as the ratio between the displacement and the corresponding accuracy of the point displacement. Usually, the obtained value of the test is compared with a factor of 3.5 or more, which represents a rough estimate (Savšek-Safić et al., 2003). Several deformation analysis approaches are known in geodesy: Hanover, Delft, Karlsruhe, etc. (Mihailović and Aleksić, 1994). The essence of such approaches lies in assessing the statistical significance of displacement based on multiple periodic measurements, under the assumptions regarding the actual risk of rejecting the null hypothesis and the associated distribution function of the chosen test statistic. Different approaches do not provide a unique solution, as they rely on different test statistics. In this paper, we discuss the *Squared M_{split} estimation* approach on a selected test case and evaluate the results through a comparative analysis with the outcomes of other deformation analysis methods. Since we were unable to obtain convincing results for the 2D geodetic network by strictly following the procedures outlined in the Wisniewski (2009b, 2009c), we had to slightly rearrange the equations and apply appropriate initial values. In this paper, we present the procedure along with the adjustments we had to make. The primary objective of this paper is to compare the results obtained using the *Squared M_{split} estimation* with those from other deformation analysis procedures, following a slightly modified set of equations described in the subsequent sections.

The *Squared M_{split} estimation* has already been applied in deformation analysis by various authors, primarily in one-dimensional (1D) leveling networks (Duchnowski and Wiśniewski, 2011, 2012; Duchnowski and Wyszkowska, 2022; Wiśniewski, 2009b, 2009c, 2010; Wiśniewski, Duchnowski and Dumalski, 2019; Wiśniewski and Zienkiewicz, 2016, 2020, 2021; Wyszkowska and Duchnowski, 2019, 2020; Zienkiewicz, 2015, 2019, 2022; Zienkiewicz and Baryla, 2015; Zienkiewicz and Dąbrowski, 2023; Zienkiewicz, Hejbudzka and Dumalski, 2017). Some researchers have also utilized the *Squared M_{split} estimation* for the deformation analysis of two-dimensional (2D) geodetic networks. For example, Zienkiewicz (2019) presented the problem of robustness of the proposed calculation strategy against gross errors occurring in the observations. Duchnowski and Wyszkowska (2022) analysed unstable object points during measurements – deformation analysis based on pseudo epoch approach. Novel (2019) applied the *Squared* $M_{split(q)}$ S-transformation of control network deformations).

2 SQUARED M_{Split} ESTIMATION

The estimation of point displacements by the *Squared* M_{split} *estimation* is an extension of the maximum likelihood method. The *Squared* M_{split} *estimation* assumes that a classical functional model can be divided into q competitive models (Wisniewski, 2009a, 2009b, 2010). Observations in each individual model thus represent a set of random variables (parameters), which may differ from one another. In our application of *Squared* M_{split} *estimation*, there is an assumption of the split of the classical functional model into two competitive functional models. The mentioned feature is also considered when solving individual geodetic problems in the field of robust transformation, deformation analysis and robust parameter estimation (Wiśniewski, Duchnowski and Dumalski, 2019). Wiśniewski (2009a, 2009b, 2010) has shown that the *Squared* M_{split} *estimation* is an alternative approach to robust methods. The approach is used both in leveling geodetic networks and in horizontal geodetic networks.

In the presented test case, we consider a horizontal geodetic network. The equations in our study are taken from already published Wiśniewski, Duchnowski and Dumalski (2019); Wiśniewski (2009a, 2009b, 2010); Wyszkowska and Duchnowski (2020).

The measurements and unknowns are related by mathematical relationships, which are generally nonlinear (e.g., Ghilani, 2010, p. 189-195; Leick, 1980, p. 51-68; Leick, Rapoport and Tatarnikov, 2015, p. 17-31; Ogundare, 2019, p. 179-191):

$$\hat{\mathbf{y}} = \boldsymbol{f}(\hat{\mathbf{x}}) \text{ or } \mathbf{y} - \mathbf{v} = \boldsymbol{f}(\mathbf{x_0} + \mathbf{x}), \tag{1}$$

where:

$\hat{\mathbf{y}}$ … vector of adjusted observations,

$\hat{\mathbf{x}}$ … vector of adjusted parameters,

$\boldsymbol{f}(\cdot)$… nonlinear mathematical functions,

$\mathbf{y}$ … vector of observations,

$\mathbf{v}$ … vector of residuals,

$\mathbf{x_0}$ … vector of approximate values of the parameters,

$\mathbf{x}$ … vector of parameters.

It should be noted that in Equation (1), we have intentionally written the difference on the left-hand side to ensure consistency with the derived equations of the *Squared* M_{split} *estimation*. In the literature (e.g., Ghilani, 2010; Leick, 1980; Leick, Rapoport and Tatarnikov, 2015; Ogundare, 2019), the sum is typically written on the left-hand side of Equation (1).

By expanding the nonlinear Equation (1) into a Taylor series around the approximate values of unknowns $\mathbf{x_0}$, we obtain a linearized form of Equation (1):

$$\mathbf{y}-\mathbf{v}=\boldsymbol{f}(\mathbf{x_0})+\left.\frac{\partial f}{\partial \mathbf{x}}\right|_{\mathbf{x_0}}\mathbf{x}\text{ or }\mathbf{f}=\mathbf{A}\mathbf{x}+\mathbf{v}, \tag{2}$$

where:

$\boldsymbol{f}(\mathbf{x_0})=\mathbf{y_0}$ … vector of the value of the observations as computed from the approximate parameters $\mathbf{x_0}$,

$\mathbf{A}=\left.\frac{\partial f}{\partial \mathbf{x}}\right|_{\mathbf{x_0}}$ … design matrix, matrix of known coefficients,

$\mathbf{f}=\mathbf{y}-\boldsymbol{f}(\mathbf{x_0})=\mathbf{y}-\mathbf{y_0}$ … misclosure vector – the discrepancy between the observations $\mathbf{y}$ and value of the observations $\mathbf{y_0}$, as computed from the approximate parameters $\mathbf{x_0}$.

The considered approach divides the basic Equation (2) into q parts, Equation (3), where q is the number of observation sets (in previous studies: Wiśniewski, Duchnowski and Dumalski, 2019; Wiśniewski, 2009a, 2009b, 2010; Wyszkowska and Duchnowski, 2020, vector of observations is denoted as $\mathbf{y}$, whereas in our study, based on Equation (2), we denote the vector of observations $\mathbf{f}$, which is valid up to Equation (59)):

$$\mathbf{f}=\mathbf{A}\mathbf{x}+\mathbf{v}\xrightarrow{\textbf{split}}\begin{cases}\mathbf{f}=\mathbf{A}\mathbf{x}_1+\mathbf{v}_1\\ \quad\vdots\\ \mathbf{f}=\mathbf{A}\mathbf{x}_q+\mathbf{v}_q\end{cases}. \tag{3}$$

For ease of discussion, it is assumed that there are two groups of field observations (measurement epochs), which are denoted by α (1st epoch set) and β (2nd epoch set). The Equation (3) thus becomes:

$$\begin{aligned}\mathbf{f}&=\mathbf{A}\mathbf{x}_\alpha+\mathbf{v}_\alpha\\ \mathbf{f}&=\mathbf{A}\mathbf{x}_\beta+\mathbf{v}_\beta\end{aligned}. \tag{4}$$

With the considered approach, it is necessary to calculate the estimated parameters $\hat{\mathbf{x}}_\alpha$ and $\hat{\mathbf{x}}_\beta$ in each iteration for each measurement in the vector of deviations $\mathbf{f}$, as well as the corresponding corrections v $\hat{\mathbf{v}}_\alpha$ and $\hat{\mathbf{v}}_\beta$, for which the function is treated in the form (Wiśniewski, Duchnowski and Dumalski, 2019; Wiśniewski, 2009a, 2009b, 2010; Wyszkowska and Duchnowski, 2020):

$$\min_{\mathbf{x}_\alpha,\mathbf{x}_\beta}\varphi\left(\mathbf{f};\mathbf{x}_\alpha,\mathbf{x}_\beta\right)=\varphi\left(\mathbf{f};\hat{\mathbf{x}}_\alpha,\hat{\mathbf{x}}_\beta\right), \tag{5}$$

where:

$$\aleph\left(\mathbf{f};\mathbf{x}_\alpha,\mathbf{x}_\beta\right)=\sum_{i=1}^{n}{}_{\alpha}\left(f_i;\mathbf{x}_\alpha\right){}_{\beta}\left(f_i;\mathbf{x}_\beta\right)=\left[\boldsymbol{\rho}_\alpha\left(\mathbf{f};\mathbf{x}_\alpha\right)\right]^{\mathrm{T}}\boldsymbol{\rho}_\beta\left(\mathbf{f};\mathbf{x}_\beta\right) \tag{6}$$

and $i=1,\ldots,n$, where n is the number of measurements in all epochs combined.

If the functions $\rho_\alpha(y_i;\mathbf{x}_\alpha)$ and $\rho_\beta(y_i;\mathbf{x}_\beta)$ are convex and their second-order derivatives exist, Newton's method can be used to solve the problem of Equation (5) (Teunissen, 1990; Wiśniewski, 2009a, 2009b). The parameters $\hat{\mathbf{x}}_\alpha$ and $\hat{\mathbf{x}}_\beta$ are the solutions of the considered method when the gradient of the Equation (6) is equal to zero, i.e. (Wiśniewski, Duchnowski and Dumalski, 2019; Wiśniewski, 2009a, 2009b):

$$\mathbf{g}_{\alpha}(\mathbf{x}_{\alpha},\mathbf{x}_{\beta})\Big|_{\substack{\mathbf{x}_{\alpha}=\hat{\mathbf{x}}_{\alpha}\\ \mathbf{x}_{\beta}=\hat{\mathbf{x}}_{\beta}}} = \frac{\partial}{\partial \mathbf{x}_{\alpha}}\varphi(\mathbf{f};\mathbf{x}_{\alpha},\mathbf{x}_{\beta}) = \mathbf{0} \text{ and} \tag{7}$$

$$\mathbf{g}_{\beta}(\mathbf{x}_{\alpha},\mathbf{x}_{\beta})\Big|_{\substack{\mathbf{x}_{\alpha}=\hat{\mathbf{x}}_{\alpha}\\ \mathbf{x}_{\beta}=\hat{\mathbf{x}}_{\beta}}} = \frac{\partial}{\partial \mathbf{x}_{\beta}}\varphi(\mathbf{f};\mathbf{x}_{\alpha},\mathbf{x}_{\beta}) = \mathbf{0}. \tag{8}$$

The partial derivatives in Equations (7) and (8) can be written as (Wiśniewski, 2009a, 2010):

$$\frac{\partial}{\partial \mathbf{x}_{\alpha}}\varphi(\mathbf{f};\mathbf{x}_{\alpha},\mathbf{x}_{\beta}) = \frac{\partial \mathbf{v}_{\alpha}}{\partial \mathbf{x}_{\alpha}}\frac{\partial}{\partial \mathbf{v}_{\alpha}}\varphi(\mathbf{f};\mathbf{x}_{\alpha},\mathbf{x}_{\beta}) = \frac{\partial \mathbf{v}_{\alpha}}{\partial \mathbf{x}_{\alpha}}\left[\rho_{\beta}(v_{1\beta})\frac{\partial \rho_{\alpha}(v_{1\alpha})}{\partial v_{1\alpha}},\ldots,\rho_{\beta}(v_{n\beta})\frac{\partial \rho_{\alpha}(v_{n\alpha})}{\partial v_{n\alpha}}\right]^{\mathrm{T}} \text{ and} \tag{9}$$

$$\frac{\partial}{\partial \mathbf{x}_{\beta}}\varphi(\mathbf{f};\mathbf{x}_{\alpha},\mathbf{x}_{\beta}) = \frac{\partial \mathbf{v}_{\beta}}{\partial \mathbf{x}_{\beta}}\frac{\partial}{\partial \mathbf{v}_{\beta}}\varphi(\mathbf{f};\mathbf{x}_{\alpha},\mathbf{x}_{\beta}) = \frac{\partial \mathbf{v}_{\beta}}{\partial \mathbf{x}_{\beta}}\left[\rho_{\alpha}(v_{1\alpha})\frac{\partial \rho_{\beta}(v_{1\beta})}{\partial v_{1\beta}},\ldots,\rho_{\alpha}(v_{n\alpha})\frac{\partial \rho_{\beta}(v_{n\beta})}{\partial v_{n\beta}}\right]^{\mathrm{T}}. \tag{10}$$

In Equations (9) and (10), the elements in the vector can be simplified as follows (Wiśniewski, 2009a):

$$\boldsymbol{\rho}_{\alpha}(\mathbf{f};\mathbf{x}_{\alpha}) \aleph \left[\rho_{\alpha}(f_1;\mathbf{x}_{\alpha}),\ldots,\rho_{\alpha}(f_n;\mathbf{x}_{\alpha})\right]^{\mathrm{T}} \quad \left[\rho_{\alpha}(v_{1\alpha}),\ldots,\rho_{\alpha}(v_{n\alpha})\right]^{\mathrm{T}} \quad \boldsymbol{\rho}_{\alpha}(\mathbf{v}_{\alpha}) \text{ and} \tag{11}$$

$$\boldsymbol{\rho}_{\beta}(\mathbf{f};\mathbf{x}_{\beta}) \aleph \left[\rho_{\beta}(f_1;\mathbf{x}_{\beta}),\ldots,\rho_{\beta}(f_n;\mathbf{x}_{\beta})\right]^{\mathrm{T}} \quad \left[\rho_{\beta}(v_{1\beta}),\ldots,\rho_{\beta}(v_{n\beta})\right]^{\mathrm{T}} \quad \boldsymbol{\rho}_{\beta}(\mathbf{v}_{\beta}). \tag{12}$$

Next, the terms $\rho_{\alpha}(\mathbf{v}_{\alpha})$ and $\rho_{\beta}(\mathbf{v}_{\beta})$ from Equations (11) and (12) are converted into a diagonal matrix (Wiśniewski, 2009a, 2010):

$$\mathrm{diag}\{\boldsymbol{\rho}_{\alpha}(\mathbf{v}_{\alpha})\} = \mathrm{diag}\{\rho_{\alpha}(v_{1\alpha}),\ldots,\rho_{\alpha}(v_{n\alpha})\} \text{ and} \tag{13}$$

$$\mathrm{diag}\{\boldsymbol{\rho}_{\beta}(\mathbf{v}_{\beta})\} = \mathrm{diag}\{\rho_{\beta}(v_{1\beta}),\ldots,\rho_{\beta}(v_{n\beta})\}. \tag{14}$$

In addition, the following applies (Wiśniewski, 2009a, 2010):

$$\left[\frac{\partial \rho_{\alpha}(v_{1\alpha})}{\partial v_{1\alpha}},\ldots,\frac{\partial \rho_{\alpha}(v_{n\alpha})}{\partial v_{n\alpha}}\right]^{\mathrm{T}} = \frac{\partial \boldsymbol{\rho}_{\alpha}(\mathbf{v}_{\alpha})}{\partial \mathbf{v}_{\alpha}} = \breve{\ }_{M\alpha}(\quad_{\alpha}) \text{ and} \tag{15}$$

$$\left[\frac{\partial \rho_{\beta}(v_{1\beta})}{\partial v_{1\beta}},\ldots,\frac{\partial \rho_{\beta}(v_{n\beta})}{\partial v_{n\beta}}\right]^{\mathrm{T}} = \frac{\partial \boldsymbol{\rho}_{\beta}(\mathbf{v}_{\beta})}{\partial \mathbf{v}_{\beta}} = \mathbf{g}_{M\beta}(\mathbf{v}_{\beta}), \tag{16}$$

$$\frac{\partial \mathbf{v}_{\alpha}}{\partial \mathbf{x}_{\alpha}} = \frac{\partial}{\partial \mathbf{x}_{\alpha}}(\mathbf{f}-\mathbf{A}\mathbf{x}_{\alpha}) = -\mathbf{A}^{\mathrm{T}} \text{ and} \tag{17}$$

$$\frac{\partial \mathbf{v}_{\beta}}{\partial \mathbf{x}_{\beta}} = \frac{\partial}{\partial \mathbf{x}_{\beta}}\breve{\ }\mathbf{f}-\mathbf{A}\mathbf{x}_{\beta} \quad = -\mathbf{A}^{\mathrm{T}} \tag{18}$$

The gradients $\mathbf{g}_{\alpha}(\hat{\mathbf{x}}_{\alpha}, \hat{\mathbf{x}}_{\beta})$ and $\mathbf{g}_{\beta}(\hat{\mathbf{x}}_{\alpha}, \hat{\mathbf{x}}_{\beta})$ in Equations (7) and (8) are expressed by considering Equations (13) – (18) (Wiśniewski, 2009a, 2009b, 2010):

$$\mathbf{g}_{\alpha}(\mathbf{x}_{\alpha},\mathbf{x}_{\beta}) \aleph \frac{\partial}{\partial \mathbf{x}_{\alpha}}\varphi(\mathbf{f};\mathbf{x}_{\alpha},\mathbf{x}_{\beta}) \qquad \mathbf{A}^{\mathrm{T}}\mathrm{diag}\{\boldsymbol{\rho}_{\beta}(\mathbf{v}_{\beta})\}\,\mathbf{g}_{M\alpha}(\mathbf{v}_{\alpha}) \text{ and} \tag{19}$$

$$\mathbf{g}_{\beta}(\mathbf{x}_{\alpha},\mathbf{x}_{\beta}) \aleph \frac{\partial}{\partial \mathbf{x}_{\beta}}\varphi(\mathbf{f};\mathbf{x}_{\alpha},\mathbf{x}_{\beta}) \qquad \mathbf{A}^{\mathrm{T}}\mathrm{diag}\{\boldsymbol{\rho}_{\alpha}(\mathbf{v}_{\alpha})\}\,\mathbf{g}_{M\beta}(\mathbf{v}_{\beta}). \tag{20}$$

Since this is the *Squared* M_{split} *estimation*, Equations (11) and (12) should be reformulated accordingly (Wiśniewski, 2009b, 2010):

$$\rho_\alpha(f_i;\mathbf{x}_\alpha)=\rho_\alpha(v_{i\alpha})=\breve{v}_{i\alpha} \;\rightarrow\; \mathrm{diag}\{\boldsymbol{\rho}_\alpha(\mathbf{v}_\alpha)\}=\mathrm{diag}\{v_{1\alpha},\ldots,v_{n\alpha}\}=\mathbf{w}_\beta(\mathbf{v}_\alpha) \text{ and} \tag{21}$$

$$\rho_\beta(f_i;\mathbf{x}_\beta)=\rho_\beta(v_{i\beta})=\breve{v}_{i\beta} \;\rightarrow\; \mathrm{diag}\{\boldsymbol{\rho}_\beta(\mathbf{v}_\beta)\}=\mathrm{diag}\{v_{1\beta},\ldots,v_{n\beta}\}=\mathbf{w}_\alpha(\mathbf{v}_\beta), \tag{22}$$

$$\breve{}_{M\alpha}(\ _\alpha)=2\left[v_{1\alpha},\ldots,v_{n\alpha}\right]^{\mathrm{T}}=2\ _\alpha \text{ and} \tag{23}$$

$$\breve{}_{M\beta}(\ _\beta)=2\left[v_{1\beta},\ldots,v_{n\beta}\right]^{\mathrm{T}}=2\ _\beta. \tag{24}$$

Based on Equations (21) – (24), the gradients $\mathbf{g}_\alpha(\mathbf{x}_\alpha, \mathbf{x}_\beta)$ and $\mathbf{g}_\beta(\mathbf{x}_\alpha, \mathbf{x}_\beta)$ are written in the final form (Wiśniewski, 2009a, 2009b, 2010):

$$\mathbf{g}_\alpha(\mathbf{x}_\alpha,\mathbf{x}_\beta)=-\mathbf{A}^{\mathrm{T}}\mathrm{diag}\{\boldsymbol{\rho}_\beta(\mathbf{v}_\beta)\}\mathbf{g}_{M\alpha}(\mathbf{v}_\alpha)=-2\mathbf{A}^{\mathrm{T}}\mathbf{w}_\alpha(\mathbf{v}_\beta)\mathbf{v}_\alpha \text{ and} \tag{25}$$

$$\mathbf{g}_\beta(\mathbf{x}_\alpha,\mathbf{x}_\beta)=-\mathbf{A}^{\mathrm{T}}\mathrm{diag}\{\boldsymbol{\rho}_\alpha(\mathbf{v}_\alpha)\}\mathbf{g}_{M\beta}(\mathbf{v}_\beta)=-2\mathbf{A}^{\mathrm{T}}\mathbf{w}_\beta(\mathbf{v}_\alpha)\mathbf{v}_\beta. \tag{26}$$

Newton's method is one of the iterative methods used to approximate roots (or zeros) of a real-valued function. Newton developed a method of solving a nonlinear equation from the secant method and the finite differences method, and Raphson then simplified it and wrote it in the form that is known today. Therefore, the method is called Newton's or Newton-Raphson's. Simpson adapted the algorithm a few years later to solve a system of nonlinear equations (Močnik, 2022). The goal of the method is to calculate a series of approximations from the initial guess using an iteration process, which has the zero of the function as a limit. It must be understood that the choice of the initial estimate of the root is crucial because the method will diverge if the initial value is chosen inappropriately. In the case of a good approximation, it will converge to a certain zero, although we have no control over which zero the method will converge to. Newton's method can be iteratively written as (Močnik, 2022):

$$\breve{}_{j+1} = \ _j \quad \frac{f(x_j)}{\breve{}(\ _j)}, \qquad 0,1,\ldots \tag{27}$$

The method can also be derived analytically. We can approximate the function f by a Taylor series at the approximation x_j:

$$f(x_j+h)=f(x_j)+f'(x_j)h+\frac{1}{2!}f''(x_j)h^2+\ldots \tag{28}$$

If we transfer a function of one variable into a multivariable function, the linear part of the Taylor series takes the form

$$f(x_{\cdot}\ \aleph h) \quad f(x\ ) \quad \mathbf{J}(x\ )h, \tag{29}$$

where x_j and h are vectors with the dimension $n \times 1$, and $\mathbf{J}$ is the Jacobian matrix of the mapping f. Newton's method for the multidimensional case therefore has a rule

$$x_{j+1}=x_j-\mathbf{J}^{-1}(x_j)f(x_j). \tag{30}$$

The Jacobian matrix represents a matrix that consists of first order partial derivatives. To solve the problem, we need the Hessian matrix, a square matrix of second-order partial derivatives. It holds that the Jacobian matrix of the gradient of the function f, denoted by ∇f, is equal to the Hessian matrix:

$$\mathbf{H}(x)=\mathbf{J}(\nabla f). \tag{31}$$

In the case of the *Squared* M_{split} *estimation* the Jacobian matrix is represented by the gradient of the Equation (7) and (8). The Hesse matrix is thus obtained by extracting the Equation (6) twice, or extracting the gradient of the Equation (7) and (8) (Wiśniewski, 2009a, 2009b, 2010):

$$\mathbf{H}_{\alpha}(\mathbf{x}_{\alpha},\mathbf{x}_{\beta}) = \frac{\partial^2}{\partial \mathbf{x}_{\alpha}\partial \mathbf{x}_{\alpha}^{\mathrm{T}}}\varphi(\mathbf{f};\mathbf{x}_{\alpha},\mathbf{x}_{\beta}) = \frac{\partial}{\partial \mathbf{x}_{\alpha}^{\mathrm{T}}}\mathbf{g}_{\alpha}(\mathbf{x}_{\alpha},\mathbf{x}_{\beta}) \text{ and} \tag{32}$$

$$\mathbf{H}_{\beta}(\mathbf{x}_{\alpha},\mathbf{x}_{\beta}) = \frac{\partial^2}{\partial \mathbf{x}_{\beta}\partial \mathbf{x}_{\beta}^{\mathrm{T}}}\varphi(\mathbf{f};\mathbf{x}_{\alpha},\mathbf{x}_{\beta}) = \frac{\partial}{\partial \mathbf{x}_{\beta}^{\mathrm{T}}}\mathbf{g}_{\beta}(\mathbf{x}_{\alpha},\mathbf{x}_{\beta}). \tag{33}$$

According to Equation (19) and (20), we have (Wiśniewski, 2009a, 2009b, 2010):

$$\frac{\partial}{\partial \mathbf{x}_{\alpha}^{\mathrm{T}}}\mathbf{g}_{\alpha}(\mathbf{x}_{\alpha},\mathbf{x}_{\beta}) = -\mathbf{A}^{\mathrm{T}}\operatorname{diag}\{\boldsymbol{\rho}_{\beta}(\mathbf{v}_{\beta})\}\frac{\partial \mathbf{g}_{M\alpha}(\mathbf{v}_{\alpha})}{\partial \mathbf{v}_{\alpha}^{\mathrm{T}}}\frac{\partial \mathbf{v}_{\alpha}}{\partial \mathbf{x}_{\alpha}^{\mathrm{T}}} \text{ and} \tag{34}$$

$$\frac{\partial}{\partial \mathbf{x}_{\beta}^{\mathrm{T}}}\mathbf{g}_{\beta}(\mathbf{x}_{\alpha},\mathbf{x}_{\beta}) = -\mathbf{A}^{\mathrm{T}}\operatorname{diag}\{\boldsymbol{\rho}_{\alpha}(\mathbf{v}_{\alpha})\}\frac{\partial \mathbf{g}_{M\beta}(\mathbf{v}_{\beta})}{\partial \mathbf{v}_{\beta}^{\mathrm{T}}}\frac{\partial \mathbf{v}_{\beta}}{\partial \mathbf{x}_{\beta}^{\mathrm{T}}}, \tag{35}$$

where:

$$\frac{\partial \mathbf{g}_{M\alpha}(\mathbf{v}_{\alpha})}{\partial \mathbf{v}_{\alpha}^{\mathrm{T}}} = \mathbf{H}_{M\alpha}(\mathbf{v}_{\alpha}) \text{ and} \tag{36}$$

$$\frac{\partial \mathbf{g}_{M\beta}(\mathbf{v}_{\beta})}{\partial \mathbf{v}_{\beta}^{\mathrm{T}}} = \mathbf{H}_{M\beta}(\mathbf{v}_{\beta}) \tag{37}$$

$$\frac{\partial \mathbf{v}_{\alpha}}{\partial \mathbf{x}_{\alpha}^{\mathrm{T}}} = \frac{\partial}{\partial \mathbf{x}_{\alpha}^{\mathrm{T}}}\left(\mathbf{f}-\mathbf{A}\mathbf{x}_{\alpha}\right) = -\mathbf{A} \text{ and} \tag{38}$$

$$\frac{\partial \mathbf{v}_{\beta}}{\partial \mathbf{x}_{\beta}^{\mathrm{T}}} = \frac{\partial}{\partial \mathbf{x}_{\beta}^{\mathrm{T}}}\left(\mathbf{f}-\mathbf{A}\mathbf{x}_{\beta}\right) = -\mathbf{A}. \tag{39}$$

According to Equations (32) – (39), we transform the Hessian matrix into (Wiśniewski, 2009a, 2009b):

$$\mathbf{H}_{\alpha}(\mathbf{x}_{\alpha},\mathbf{x}_{\beta}) = -\mathbf{A}^{\mathrm{T}}\operatorname{diag}\{\boldsymbol{\rho}_{\beta}(\mathbf{v}_{\beta})\}\frac{\partial \mathbf{g}_{M\alpha}(\mathbf{v}_{\alpha})}{\partial \mathbf{v}_{\alpha}^{\mathrm{T}}}\frac{\partial \mathbf{v}_{\alpha}}{\partial \mathbf{x}_{\alpha}^{\mathrm{T}}} = \mathbf{A}^{\mathrm{T}}\operatorname{diag}\{\boldsymbol{\rho}_{\beta}(\mathbf{v}_{\beta})\}\mathbf{H}_{M\alpha}(\mathbf{v}_{\alpha})\mathbf{A} \text{ and} \tag{40}$$

$$\mathbf{H}_{\beta}(\mathbf{x}_{\alpha},\mathbf{x}_{\beta}) = -\mathbf{A}^{\mathrm{T}}\operatorname{diag}\{\boldsymbol{\rho}_{\alpha}(\mathbf{v}_{\alpha})\}\frac{\partial \mathbf{g}_{M\beta}(\mathbf{v}_{\beta})}{\partial \mathbf{v}_{\beta}^{\mathrm{T}}}\frac{\partial \mathbf{v}_{\beta}}{\partial \mathbf{x}_{\beta}^{\mathrm{T}}} = \mathbf{A}^{\mathrm{T}}\operatorname{diag}\{\boldsymbol{\rho}_{\alpha}(\mathbf{v}_{\alpha})\}\mathbf{H}_{M\beta}(\mathbf{v}_{\beta})\mathbf{A}. \tag{41}$$

From Equations (23) and (24), the following can be concluded (Wiśniewski, 2009a, 2009b):

$$\mathbf{H}_{M\alpha}(\mathbf{v}_{\alpha}) = \frac{\partial \mathbf{g}_{M\alpha}(\mathbf{v}_{\alpha})}{\partial \mathbf{v}_{\alpha}^{\mathrm{T}}} = \frac{\partial(2\mathbf{v}_{\alpha})}{\partial \mathbf{v}_{\alpha}^{\mathrm{T}}} = 2\mathbf{I} \text{ and} \tag{42}$$

$$\mathbf{H}_{M\beta}(\mathbf{v}_{\beta}) = \frac{\partial \mathbf{g}_{M\beta}(\mathbf{v}_{\beta})}{\partial \mathbf{v}_{\beta}^{\mathrm{T}}} = \frac{\partial(2\mathbf{v}_{\beta})}{\partial \mathbf{v}_{\beta}^{\mathrm{T}}} = 2\mathbf{I}, \tag{43}$$

where **I** is an identity matrix.

The final form of the Hessian matrix is (Wiśniewski, 2009a, 2009b, 2010):

$$\mathbf{H}_{\alpha}(\mathbf{x}_{\alpha},\mathbf{x}_{\beta}) = \mathbf{A}^{\mathrm{T}}\operatorname{diag}\{\boldsymbol{\rho}_{\beta}(\mathbf{v}_{\beta})\}\mathbf{H}_{M\alpha}(\mathbf{v}_{\alpha})\mathbf{A} = 2\mathbf{A}^{\mathrm{T}}\mathbf{w}_{\alpha}(\mathbf{v}_{\beta})\mathbf{A} = \mathbf{H}_{\alpha}(\mathbf{x}_{\beta}) \text{ and} \tag{44}$$

$$\mathbf{H}_{\beta}(\mathbf{x}_{\alpha},\mathbf{x}_{\beta}) \aleph \mathbf{A}^{\mathrm{T}}\mathrm{diag}\{\boldsymbol{\rho}_{\alpha}(\mathbf{v}_{\alpha})\}\mathbf{H}_{M\beta}(\mathbf{v}_{\beta})\mathbf{A} \quad 2\mathbf{A}^{\mathrm{T}}\mathbf{w}_{\beta}(\mathbf{v}_{\alpha})\mathbf{A} \quad \mathbf{H}_{\beta}(\mathbf{x}_{\alpha}). \tag{45}$$

Based on the equations and derivations presented so far and considering Equation (30), we can proceed to the final solution of the *Squared M_{split} estimation*, or to the iterative computational procedure (Wiśniewski, 2009a, 2009b, 2010; Wyszkowska and Duchnowski, 2020):

$$\breve{\ }_{\aleph} = \quad ^{-1} + \Delta \quad ; j = 1,\ldots,k \text{ and} \tag{46}$$

$$\breve{\ }_{\aleph} \aleph \quad ^{-1} + \quad ; j \quad 1,\ldots,k, \tag{47}$$

where:

k … number of iterations,

$$\begin{aligned} \aleph\mathbf{x}_{\alpha}^{j} \quad & \left\{\mathbf{H}_{\alpha}\left(\mathbf{x}_{\alpha}^{j-1},\mathbf{x}_{\beta}^{j-1}\right)\right\}^{-1}\mathbf{g}_{\alpha}\left(\mathbf{x}_{\alpha}^{j-1},\mathbf{x}_{\beta}^{j-1}\right) \\ &= -\left\{\mathbf{H}_{\alpha}\left(\mathbf{x}_{\beta}^{\aleph}\right)\right\}^{-1}\mathbf{g}_{\alpha}\left(\mathbf{x}_{\alpha} \quad ,\mathbf{x}_{\beta} \quad \right) \\ &= \left\{\mathbf{A}^{\mathrm{T}}\mathrm{diag}\left\{\boldsymbol{\rho}_{\beta}\left(\mathbf{v}_{\beta}^{j-1}\right)\right\}\mathbf{H}_{M\alpha}\left(\mathbf{v}_{\alpha}^{j-1}\right)\mathbf{A}\right\}^{-1}\mathbf{A}^{\mathrm{T}}\mathrm{diag}\left\{\boldsymbol{\rho}_{\beta}\left(\mathbf{v}_{\beta}^{j-1}\right)\right\}\mathbf{g}_{M\alpha}\left(\mathbf{v}_{\alpha}^{j-1}\right) \\ &= \left\{\mathbf{A}^{\mathrm{T}}\mathbf{w}_{\alpha}\left(\mathbf{v}_{\beta}^{\aleph}\right)2\mathbf{A}\right\}^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{w}_{\alpha}\left(\mathbf{v}_{\beta}^{\ 1}\right)2\mathbf{v}_{\alpha}^{\ 1} \\ &= \left\{\mathbf{A}^{\mathrm{T}}\mathbf{w}_{\alpha}\left(\mathbf{v}_{\beta}^{\aleph}\right)\mathbf{A}\right\}^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{w}_{\alpha}\left(\mathbf{v}_{\beta}^{\ 1}\right)\mathbf{v}_{\alpha}^{\ 1}, \end{aligned} \tag{48}$$

$$\mathbf{w}_{\alpha}\left(\mathbf{v}_{\beta}^{\aleph}\right) = \mathrm{diag}\left\{\left(\mathrm{v}_{1\beta} \quad \right)^{2},\ldots,\left(\mathrm{v}_{n\beta} \quad \right)^{2}\right\}, \tag{49}$$

$$\mathbf{v}_{\alpha}^{j} = \mathbf{f} - \mathbf{A}\mathbf{x}_{\alpha}^{j} \text{ and} \tag{50}$$

$$\begin{aligned} \aleph\mathbf{x}_{\beta}^{j} \quad & \left\{\mathbf{H}_{\beta}\left(\mathbf{x}_{\alpha}^{j},\mathbf{x}_{\beta}^{j-1}\right)\right\}^{-1}\mathbf{g}_{\beta}\left(\mathbf{x}_{\alpha}^{j},\mathbf{x}_{\beta}^{j-1}\right) \\ &= -\left\{\mathbf{H}_{\beta}\left(\mathbf{x}_{\alpha}\right)\right\}^{-1}\mathbf{g}_{\beta}\left(\mathbf{x}_{\alpha},\mathbf{x}_{\beta}^{-1}\right) \\ &= \left\{\mathbf{A}^{\mathrm{T}}\mathrm{diag}\left\{\boldsymbol{\rho}_{\alpha}\left(\mathbf{v}_{\alpha}^{j}\right)\right\}\mathbf{H}_{M\beta}\left(\mathbf{v}_{\beta}^{j-1}\right)\mathbf{A}\right\}^{-1}\mathbf{A}^{\mathrm{T}}\mathrm{diag}\left\{\boldsymbol{\rho}_{\alpha}\left(\mathbf{v}_{\alpha}^{j}\right)\right\}\mathbf{g}_{M\beta}\left(\mathbf{v}_{\beta}^{j-1}\right) \\ &= \left\{\mathbf{A}\ \mathbf{w}_{\beta}\left(\mathbf{v}_{\alpha}\right)2\mathbf{A}\right\}^{-1}\mathbf{A}\ \ \mathbf{w}_{\beta}\left(\mathbf{v}_{\alpha}\right)2\mathbf{v}_{\beta}^{-} \\ &= \left\{\mathbf{A}\ \mathbf{w}_{\beta}\left(\mathbf{v}_{\alpha}\right)\mathbf{A}\right\}^{-1}\mathbf{A}\ \ \mathbf{w}_{\beta}\left(\mathbf{v}_{\alpha}\right)\mathbf{v}_{\beta}^{-}, \end{aligned} \tag{51}$$

$$\mathbf{w}_{\beta}\left(\mathbf{v}_{\alpha}\right) = \mathrm{diag}\left\{\left(\mathrm{v}_{1\alpha}\right)^{2},\ldots,\left(\mathrm{v}_{n\alpha}\right)^{2}\right\}, \tag{52}$$

$$\mathbf{v}_{\beta}^{j} = \mathbf{f} - \mathbf{A}\mathbf{x}_{\beta}^{j}. \tag{53}$$

Squared M_{split} estimation is an iterative process for solving a optimization problem. The initial values of the parameters $\mathbf{x}_{\alpha}^{\mathbf{0}}$ and $\mathbf{v}_{\alpha}^{\mathbf{0}}$, can be selected from the results computed using the Least Squares Method (LSM):

$$\hat{\mathbf{x}} \quad = \left(\mathbf{A}^{\mathbf{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{f}, \tag{54}$$

$$\hat{\mathbf{v}}_{\mathrm{LSM}} = \mathbf{f} - \mathbf{A}^{\mathbf{T}}\hat{\mathbf{x}}_{\mathrm{LSM}}. \tag{55}$$

In the initial step, we assume (Wiśniewski, 2009b, 2010):

$$\breve{\ }_{\alpha}^{0} = \hat{\ }\ \breve{\ } \qquad (56)$$

$$\mathbf{v}_{\alpha}^{0} = \hat{\mathbf{v}}\ , \qquad (57)$$

and compute:

$$\mathbf{x}_{\aleph}^{\breve{\ }} = \hat{\mathbf{x}}_{\mathrm{LSM}} + \left\{\mathbf{A}\ \ \mathbf{w}\ \ (\hat{\mathbf{v}}_{\mathrm{LSM}})\mathbf{A}\right\}^{-1}\mathbf{A}\ \ \mathbf{w}\ \ (\hat{\mathbf{v}}_{\mathrm{LSM}})\mathbf{f}\ \breve{\ } \qquad (58)$$

$$\mathbf{v}_{\beta}^{0} = \mathbf{f} - \mathbf{A}\mathbf{x}_{\beta}^{0}, \qquad (59)$$

where:

$$\mathbf{w}_{\beta}(\hat{\mathbf{v}}_{\mathrm{LSM}}) = \mathrm{diag}\left\{\hat{\mathbf{v}}_{1\mathrm{LSM}}^{2}, \ldots, \hat{\mathbf{v}}_{n\mathrm{LSM}}^{2}\right\}. \qquad (60)$$

All further iteration steps are calculated according to Equations (46) and (47). The stopping criterion for the iterative procedure is the convergence of the solution. The process terminates when the norms of the correction vectors for the approximate coordinates become sufficiently small (Wyszkowska and Duchnowski, 2020):

$$\left\|\aleph\mathbf{x}_{\alpha}^{j}\right\|\ \ \varepsilon\ \breve{\ } \qquad (61)$$

$$\left\|\Delta\mathbf{x}_{\beta}^{j}\right\| < \varepsilon, \qquad (62)$$

where ε is the chosen threshold for terminating the iterative process.

$\hat{\mathbf{x}}_{\alpha}^{k}$ and $\hat{\mathbf{x}}_{\beta}^{k}$, which are obtained from the last iteration step k, are therefore the final solution of the *Squared* M_{split} *estimation*. Finally, we can obtain the displacement of a single point in the considered geodetic network as:

$$\breve{\ } = \left(\hat{\ }_{\beta}^{k} - \hat{\ }_{\alpha}^{k}\right). \qquad (63)$$

The problem with this approach compared to other deformation analysis procedures is that it does not deal with a statistical test based on which we can define whether the movement is statistically significant or not. The result of the approach is only the magnitude of the point displacement. Therefore, the success in the interpretation of the results depends on the knowledge provided by geodetic experts.

3 CASE STUDY

We would like to demonstrate the effectiveness of the *Squared* M_{split} *estimation* using an example from the literature (Mihailović and Aleksić, 1994).

The simulated geodetic network (Figure 1) consists of 7 points and observed 24 horizontal directions and 24 distances. We choose σ_{Hz} = 1" as the value of a-priori variance for angular observations, and the value of a-priori variance for distances is σ_{d} = 5 mm. For both successive measurements α (1st epoch) and β (2nd epoch) the observation plan is the same. Other deformation analysis methods have also been applied to the same example of a relative geodetic network:

- Hannover (developed by H. Pelzer – Pelzer, 1971; Ambrožič, 2001),
- Karlsruhe (developed by K.R. Koch, B. Heck, E. Kuntz and B. Meier-Hirmer – Heck, Kuntz in Meier-Hirmer, 1977; Ambrožič, 2004),

- Delft (developed by J. van Mierlo and J.J. Kok – Heck et al., 1982; Marjetič, Zemljak in Ambrožič, 2013),
- Fredericton (developed by A. Chrzanowski, Y.Q. Chen and J.M Secord – Chen, Chrzanowski and Secord, 1990; Vrečko and Ambrožič, 2013),
- München (developed by W. Welsch – Welsch, 1982; Soldo and Ambrožič, 2018),
- Robust methods (iterative weight adaption developed by Y.Q. Chen – Chen, 1983; Ambrožič et al., 2019),
- Caspary deformation analysis (developed by W.F. Caspary – Caspary, 2000; Hamza, Stopar and Ambrožič, 2020).

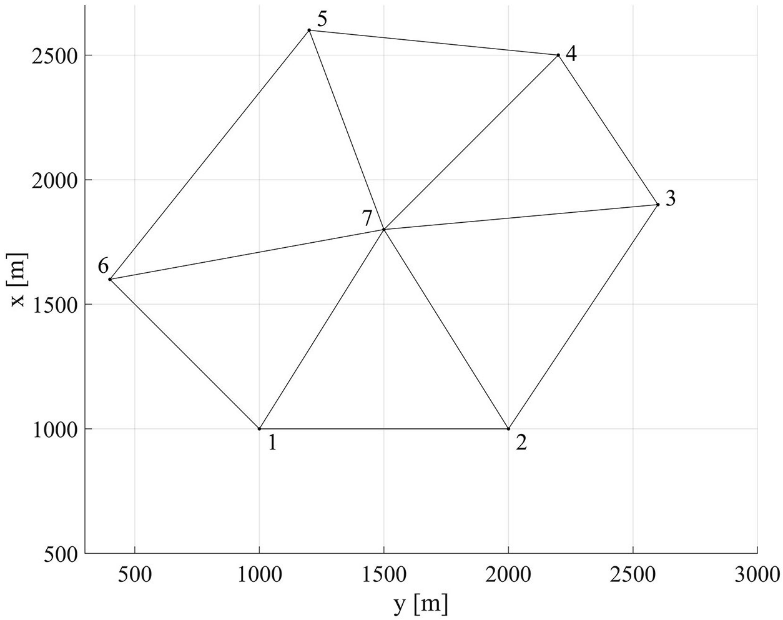

Figure 1: Geodetic network

Based on the derived equations of the *Squared M_{split} estimation* in Section 2, the input data for the calculation include the coefficient matrix **A** of the adjustment equations and the deviation vector **f**, see Equation (2).

Since we are dealing with two term epoch measurements, the matrix **A** and vector **f** must contain elements related to, for example, the directions and then the distances of the first epoch, followed by the elements related to the directions and then the distances of the second epoch. The elements of the matrix **A** and vector **f** for the measured direction are calculated, for instance, using Equations 7.51 and 7.52 (Mihailović, 1981, p. 313):

$$v_{ri} = a_{ri}x_r + b_{ri}y_r + a_{ir}x_i + b_{ir}y_i + z_r + f_{ri}, \qquad (64)$$

$$a_{ri} = -\frac{\sin n_{ri}}{s^0_{ri}}, b_{ri} = \frac{\cos n_{ri}}{s^0_{ri}}, a_{ir} = -a_{ri}, b_{ir} = -b_{ri} \ldots \text{elements of the matrix } \mathbf{A},$$

x_r, y_r, x_i, y_i and z_r … corrections to approximate values of coordinates of points *r* and *i* and corrections to orientation angle on point *r*,

$f_{ri} = n_{ri} + z_r^0 - \alpha_{ri}$ … misclosure value,

n_{ri} … approximate direction angle between points r and i,

z_r^0 … approximate orientation angle on point r,

α_{ri} … measured direction from r towards i,

S_{ri}^0 … approximate distance between r and i, calculated from the approximate coordinates of points r and i.

Elements of matrix **A** and vector **f**, for the measured distance, are computed using Equations 8.37 (Mihailović, 1981, p. 408):

$$v_{ri} = a_{ri}x_r + b_{ri}y_r + a_{ir}x_i + b_{ir}y_i + f_{ri}, \tag{65}$$

$a_{ri} = \cos n_{ri}$, $b_{ri} = \sin n_{ri}$, $a_{ir} = -a_{ri}$, $b_{ir} = -b_{ri}$ … elements of the matrix **A**,

$f_{ri} = S_{ri}^0 - S_{ri}$ … misclosure value,

S_{ri} … measured distance between r and i.

Of course, elements related to the orientation unknowns must be eliminated from the Equation (2), for example, using Gaussian elimination. Since 2D geodetic networks involve different types of measurements (directions and distances), which generally have varying levels of accuracy, we must account for weights in the *Squared* M_{split} *estimation* equations (Zienkiewicz, Hejbudzka and Dumalski, 2017; Zienkiewicz and Baryla, 2015), or use equivalent Equations (2), or homogenize the coefficient matrix **A** and the deviation vector **f**, for instance, by applying Schreiber's third rule.

The corrections of the approximate values of the coordinate unknowns $\mathbf{x}_\alpha^0$ in the initial iteration of the parameter estimation process $\hat{\mathbf{x}}_\alpha$ and $\hat{\mathbf{x}}_\beta$ of the *Squared* M_{split} *estimation* are calculated using the Least Squares Method $\hat{\mathbf{x}}_{LSM}$ according to Equation (56), and are presented in Table 1. It should be noted that these values are not the same as those listed in Table 2 in Ambrožič (2001). The values calculated here are obtained from the adjustment of both epochs simultaneously, whereas the values in Table 2 in Ambrožič (2001) were obtained from the adjustment of each epoch separately. According to Equation (58), the corrections for the approximate values of the coordinate unknowns $\mathbf{x}_\beta^0$ are also calculated and presented in Table 1.

Table 1: Corrections of the approximate values of the coordinate unknowns $\mathbf{x}_\alpha^0$ and $\mathbf{x}_\beta^0$ [m] in the initial iteration.

Point *i*	$\mathbf{x}_\alpha^0 = \hat{\mathbf{x}}_{LSM}$ using (56)		$\mathbf{x}_\beta^0$ using (58)	
	$\mathbf{x}_\alpha^0$ for y_i	$\mathbf{x}_\alpha^0$ for x_i	$\mathbf{x}_\beta^0$ for y_i	$\mathbf{x}_\beta^0$ for x_i
1	– 0.0083	– 0.0216	– 0.0182	– 0.0417
2	– 0.0142	0.0267	– 0.0311	0.0553
3	0.0143	– 0.0193	0.0305	– 0.0379
4	– 0.0004	0.0045	0.0012	0.0098
5	– 0.0048	– 0.0047	– 0.0104	– 0.0123
6	– 0.0009	– 0.0073	– 0.0024	– 0.0165
7	0.0143	0.0218	0.0304	0.0435

In the initial iteration, we calculate the corrections of the measurements $\mathbf{v}_\alpha^0 = \hat{\mathbf{v}}_{LSM}$ Equation (57) and $\mathbf{v}_\beta^0 = \mathbf{f} - \mathbf{A}\mathbf{x}_\beta^0$, using Equation (59).

The calculation of the parameters $\hat{\mathbf{x}}_\alpha$ and $\hat{\mathbf{x}}_\beta$ using the *Squared M_{split} estimation* continues with the iterative procedure as described in Equations (46) to (53). The results for $\mathbf{x}_\alpha^j$ are presented in Table 2, and the results for $\mathbf{x}_\beta^j$ are presented in Table 3. The iterative procedure is terminated when the conditions (59) and (60) are met. The threshold for stopping the iterative process was chosen at $\varepsilon = 0.001$. The conditions for terminating the iterative process were satisfied after the 8th iteration.

Table 2: Corrections of the approximate values of the coordinate unknowns $\mathbf{x}_\alpha^j$ [mm].

Coordinate	$\mathbf{x}_\alpha^0 = \hat{\mathbf{x}}_{LSM}$	$\mathbf{x}_\alpha^1$	$\mathbf{x}_\alpha^2$	$\mathbf{x}_\alpha^3$	$\mathbf{x}_\alpha^4$	$\mathbf{x}_\alpha^5$	$\mathbf{x}_\alpha^6$	$\mathbf{x}_\alpha^7$	$\mathbf{x}_\alpha^8$
y_1	– 8.3	– 3.7	– 5.0	– 5.0	– 5.0	– 4.9	– 4.9	– 4.9	– 4.9
x_1	– 21.6	– 1.5	– 0.3	0.0	0.3	0.7	1.0	1.0	1.0
y_2	– 14.2	– 0.2	0.1	0.1	– 0.0	– 0.1	– 0.1	– 0.1	– 0.1
x_2	26.7	2.1	2.3	2.4	2.6	2.7	2.7	2.7	2.7
y_3	14.3	3.2	4.0	4.4	4.9	5.6	5.9	6.0	6.0
x_3	– 19.3	– 1.0	– 1.0	– 1.2	– 1.5	– 1.9	– 2.1	– 2.1	– 2.1
y_4	– 0.4	1.3	2.0	2.5	3.1	4.0	4.5	4.5	4.5
x_4	4.5	2.9	2.5	2.4	2.1	1.8	1.7	1.7	1.7
y_5	– 4.8	1.2	– 0.2	– 1.3	– 2.5	– 4.4	– 5.6	– 5.8	– 5.8
x_5	– 4.7	– 0.2	– 0.8	– 1.0	– 1.1	– 1.5	– 1.9	– 2.0	– 2.0
y_6	– 0.9	– 3.3	– 2.3	– 2.2	– 2.2	– 2.0	– 1.9	– 1.8	– 1.8
x_6	– 7.3	– 2.9	– 3.4	– 3.4	– 3.4	– 3.3	– 3.0	– 2.9	– 2.9
y_7	14.3	1.4	1.5	1.6	1.6	1.8	2.0	2.1	2.1
x_7	21.8	0.5	0.6	0.8	1.1	1.4	1.6	1.6	1.6

Table 3: Corrections of the approximate values of the coordinate unknowns $\mathbf{x}_\beta^j$ [mm].

Coordinate	$\mathbf{x}_\beta^0$	$\mathbf{x}_\beta^1$	$\mathbf{x}_\beta^2$	$\mathbf{x}_\beta^3$	$\mathbf{x}_\beta^4$	$\mathbf{x}_\beta^5$	$\mathbf{x}_\beta^6$	$\mathbf{x}_\beta^7$	$\mathbf{x}_\beta^8$
y_1	– 18.2	– 13.3	– 13.1	– 13.2	– 13.2	– 13.3	– 13.3	– 13.3	– 13.3
x_1	– 41.7	– 41.2	– 41.7	– 41.9	– 42.3	– 42.6	– 42.7	– 42.7	– 42.7
y_2	– 31.1	– 31.3	– 30.9	– 30.9	– 30.8	– 30.8	– 30.8	– 30.8	– 30.8
x_2	55.3	53.2	52.8	52.7	52.6	52.5	52.6	52.6	52.6
y_3	30.5	26.9	26.3	25.9	25.3	24.8	24.8	24.8	24.8
x_3	– 37.9	– 36.9	– 36.6	– 36.4	– 36.0	– 35.7	– 35.7	– 35.7	– 35.7
y_4	1.2	– 1.2	– 1.8	– 2.4	– 3.1	– 3.8	– 3.9	– 3.9	– 3.9
x_4	9.8	6.8	7.1	7.3	7.6	7.7	7.8	7.7	7.7
y_5	– 10.4	– 10.2	– 9.7	– 8.7	– 7.1	– 5.5	– 5.1	– 5.0	– 5.0
x_5	– 12.3	– 11.6	– 12.0	– 11.9	– 11.6	– 11.2	– 10.9	– 10.9	– 10.9
y_6	– 2.4	0.3	0.3	0.3	0.2	– 0.0	– 0.1	– 0.1	– 0.1
x_6	– 16.5	– 13.2	– 12.1	– 11.9	– 12.1	– 12.4	– 12.7	– 12.7	– 12.7
y_7	30.4	28.8	28.9	28.9	28.8	28.5	28.4	28.4	28.4
x_7	43.5	42.9	42.5	42.2	41.9	41.7	41.7	41.7	41.7

Tables 2 and 3 indicate that the corrections of the approximate values of unknown parameters changed the most in the first iteration. The gradients for individual coordinates also underwent the most significant changes during this first iteration and approached zero in the eighth iteration. However, they did not reach exactly zero due to the conditions imposed by Equations (61) and (62). The changes in the corrections of the approximate values of unknown parameters are further illustrated in Figure 2, where the values $\mathbf{x}_\alpha^j$ were adjusted by subtracting $\mathbf{x}_\alpha^8$, and the values $\mathbf{x}_\beta^j$ were adjusted by subtracting $\mathbf{x}_\beta^8$.

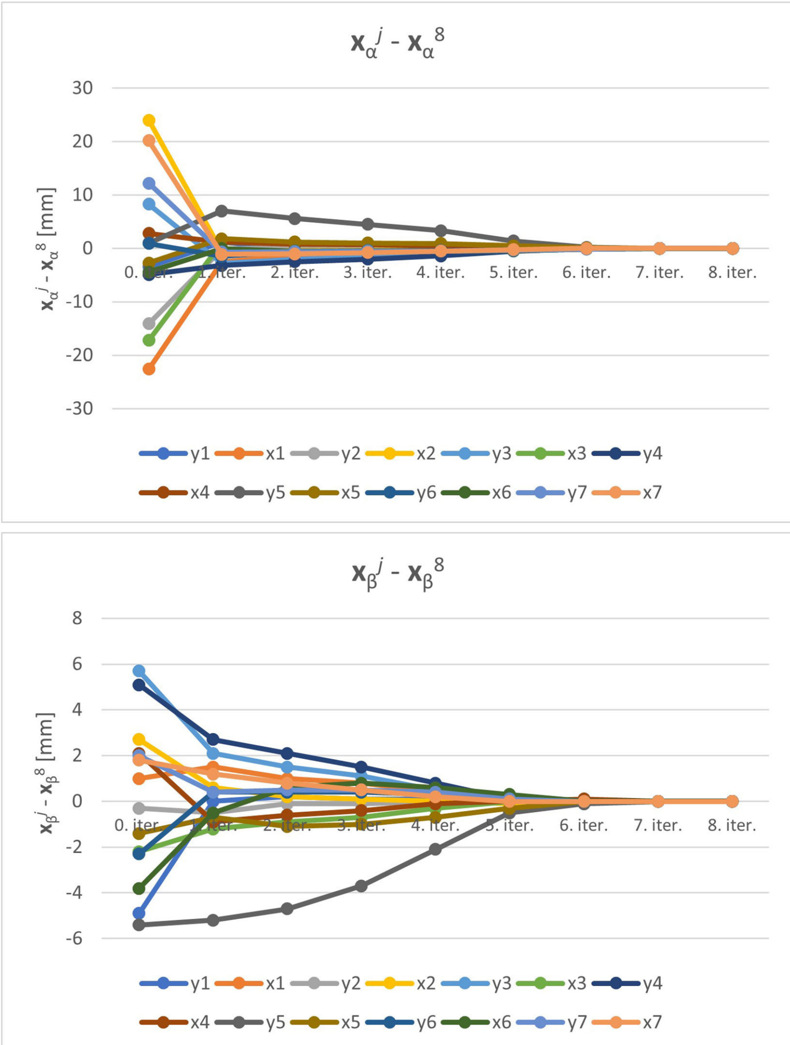

Figure 2: Changes in corrections of approximate values of unknown parameters across iterations

Finally, we calculate the displacement of a single point in the considered 2D geodetic network according to Equation (61), the results are shown in Table 4.

4 COMPARISON WITH THE RESULTS OF OTHER APPROCHES

Table 4 shows a comparison of the results of the *Squared* M_{split} *estimation* and the results of the Hannover, Karlsruhe, Delft, Fredericton, München and Caspary approaches. Compared to the other approaches, minor differences in the results of the obtained displacements are observed.

Table 4: Simulated point displacements and deformation analysis results of the Hannover, Karlsruhe, Delft, Fredericton, München, Caspary and Squared M_{split} estimation approaches.

Point		1	2	3	4	5	6	7
Simulated	d_y [mm]	– 20.0	– 30.0	25.0	0.0	0.0	0.0	25.0
	d_x [mm]	– 34.6	52.0	– 43.3	0.0	0.0	0.0	43.3
	d [mm]	40.0	60.0	50.0	0.0	0.0	0.0	50.0
	υ [°]	210	330	150	–	–	–	30
Hannover	d_y [mm]	– 19.6	– 38.7	20.6	– 4.0	– 6.4	3.3	23.6
	d_x [mm]	– 38.0	49.0	– 44.3	5.1	– 7.1	– 10.6	42.9
	d [mm]	42.8	62.4	48.9	6.5	10.0	11.1	49.0
	υ [°]	207	322	155	322	222	163	29
	Displacement	yes	yes	yes	no	no	no	yes
Karlsruhe	d_y [mm]	– 19.7	– 38.8	20.6	–	–	–	23.6
	d_x [mm]	– 38.0	49.0	– 44.4	–	–	–	42.9
	d [mm]	42.8	62.5	48.9	–	–	–	49.0
	υ [°]	207	322	155	–	–	–	29
	Displacement	yes	yes	yes	no	no	no	yes
Delft	d_y [mm]	– 19.4	– 38.1	21.4	0.7	– 0.8	0.0	24.0
	d_x [mm]	– 37.5	49.5	– 43.5	1.0	– 2.3	1.3	42.9
	d_y [mm]	42.2	62.5	48.5	1.2	2.4	1.3	49.2
	υ [°]	207	322	154	35	199	0	29
	Displacement	yes	yes	yes	no	no	no	yes
Fredericton	d_y [mm]	– 19.6	– 38.7	20.6	–	–	–	23.6
	d_x [mm]	– 38.0	49.0	– 44.3	–	–	–	42.9
	d [mm]	42.8	62.5	48.9	–	–	–	48.9
	υ [°]	207	322	155	–	–	–	29
	Displacement	yes	yes	yes	no	no	no	yes
München	d_y [mm]	– 19.5	– 38.2	21.4	0.7	– 0.8	0.0	24.0
	d_x [mm]	– 37.6	49.5	– 43.6	1.0	– 2.2	1.4	42.9
	d [mm]	42.4	62.5	48.6	1.2	2.3	1.4	49.2
	υ [°]	207	322	154	35	200	0	29
	Displacement	yes	yes	yes	no	no	no	yes
Caspary	d_y [mm]	– 19.2	– 38.4	20.8	–	–	–	23.9
	d_x [mm]	– 37.9	49.4	– 43.9	–	–	–	43.1
	d [mm]	42.5	62.5	48.6	–	–	–	49.2
	υ [°]	207	322	154	–	–	–	9
	Displacement	yes	yes	yes	no	no	no	yes
M_{Split}	d_y [mm]	**– 8.4**	**– 30.8**	**18.8**	**– 8.4**	**0.8**	**1.7**	**26.3**
	d_x [mm]	**– 43.7**	**49.9**	**– 33.5**	**6.0**	**– 8.9**	**– 9.8**	**40.1**
	d [mm]	**44.5**	**58.6**	**38.5**	**10.4**	**9.0**	**10.0**	**48.0**
	υ [°]	**191**	**328**	**151**	**306**	**175**	**170**	**33**
	Displacement	–	–	–	–	–	–	–

In Table 4, we have used notation consistent with other articles where different deformation analysis methods were examined. The notation d_y represents $\mathbf{p}_{\boldsymbol{y}} = (\hat{\mathbf{x}}_\beta^k - \hat{\mathbf{x}}_\alpha^k)_y$, Equation (63), which denotes the difference in corrections of the approximate values of unknown parameters for a specific point, calculated in the final iteration *k*. Similarly, d_x represents $\mathbf{p}_{\boldsymbol{x}} = (\hat{\mathbf{x}}_\beta^k - \hat{\mathbf{x}}_\alpha^k)_x$. The notation *d* represents the displacement of an individual point, computed as $\breve{\ } = \sqrt{\ _{y}^{2} + \ _{x}^{2}}$, while υ represents the azimuth of the displacement direction, given by $\upsilon = \text{atan}(d_y / d_x)$. The column labelled *Displacement* indicates whether a point is stable or unstable.

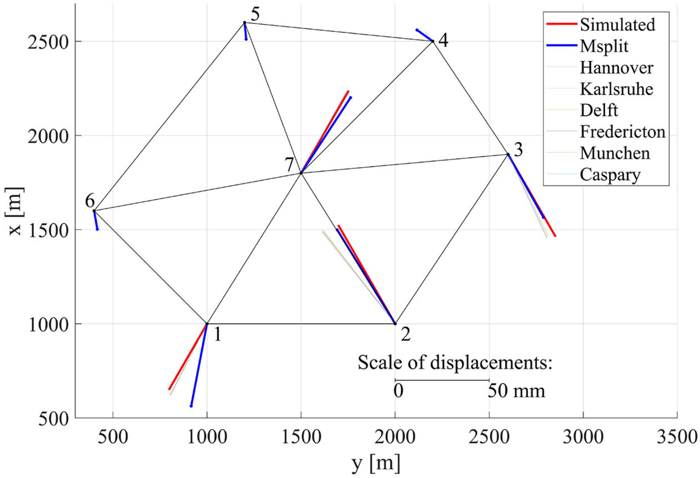

Figure 3: Plot of displacements determined by different methods of deformation analysis

Table 4 and Figure 3 shows that, for four out of seven points in the geodetic network, the discrepancies between the computed values and the simulated values are the largest when using this method, suggesting that the method is only conditionally applicable. At point 3, this method produces the largest positional discrepancy among all examined methods, reaching 11.5 mm. On the other hand, the *Squared M_{split} estimation* produced results fully consistent with other methods in identifying those points 1, 2, 3 and 7 exhibited significant movement, while points 4, 5 and 6 showed only minor displacements, which is crucial for deformation analysis.

5 CONCLUSIONS

In this paper, we describe and discuss the capabilities of *Squared M_{split} estimation* in 2D deformation analysis. Unlike previous research that has focused on leveling networks, our study concentrates on the practical application of *Squared M_{split} estimation* in angular-linear horizontal geodetic networks, supported by thorough theoretical and empirical analyses.

After deriving and presenting the equations underlying the *Squared M_{split} estimation*, we applied the computation to the same simulated 2D geodetic network as in deformation analyses performed using the Hannover, Karlsruhe, Delft, Fredericton, München, and Caspary approaches.

In the *Squared M_{split} estimation*, we first selected the initial values $\mathbf{x}_\alpha^0$ from the Least Squares Method solution $\hat{\mathbf{x}}_{\text{LSM}}$, setting $\mathbf{x}_\alpha^0 = \hat{\mathbf{x}}_{\text{LSM}}$ and $\mathbf{v}_\alpha^0 = \hat{\mathbf{v}}_{\text{LSM}}$. It was found that choosing initial values 20 times larger ($\mathbf{x}_\alpha^0 = 20 \cdot \hat{\mathbf{x}}_{\text{LSM}}$) or 1000 times smaller ($\mathbf{x}_\alpha^0 = \hat{\mathbf{x}}_{\text{LSM}}/1000$) led to the same final result, with only the number of iterations required to reach convergence differing. Subsequently, we computed $\mathbf{x}_\beta^0$ and $\mathbf{v}_\beta^0$ using Equations (58) and (59). The iterative process to determine the final corrections of the approximate values of the unknown parameters, was performed using Equations (46) through (53), yielding the final result after eight iterations. In the eighth iteration, the differences between $\mathbf{x}_\alpha^8$ and $\mathbf{x}_\beta^8$ as well as between $\mathbf{x}_\alpha^7$ and $\mathbf{x}_\beta^7$, were below the chosen convergence threshold $\varepsilon = 0.001$. Finally, we computed the displacements of individual points in the analyzed geodetic network using Equation (63) and compared the results with those from deformation analyses performed using the Hannover, Karlsruhe, Delft, Fredericton, München, and Caspary approaches, as well as with the simulated results.

From Table 4, we observe that the calculated displacements of the points are comparable to the simulated ones. Based on the results, we can conclude that the deformation analysis using the *Squared* M_{split} *estimation* is suitable for determining point displacements in a 2D geodetic network.

Compared to the other listed approaches, the main limitation of the *Squared* M_{split} *estimation* is the absence of statistical test metrics that could determine whether a detected displacement is statistically significant. However, the *Squared* M_{split} *estimation* offers advantages in that it does not require prior assumptions regarding whether points in the network are stable or unstable. It remains a valuable additional method for deformation analysis, particularly in challenging cases where point displacements in 2D networks are small.

In the future, we plan to test the method with more than two measurement epochs. Additionally, we intend to apply the method to a 3D geodetic network, where height data of the points will also be analysed.

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Kvadratna metoda M_{split} v deformacijski analizi na primeru 2D geodetske mreže

OSNOVNE INFORMACIJE O ČLANKU

GLEJ STRAN 115

1 UVOD

Pri deformacijski analizi periodično merimo geodetske mreže, s čimer odkrivamo in določamo prostorske premike točk, preko njih pa ocenjujemo premike in deformacije grajenega (hidroelektrarne, dimniki, premostitveni objekti itd.) in naravnega okolja (plazovita območja, odlagališča kamnin/zemljin itd.). Referenčne točke določajo geodetski datum mreže, ki je definiran kot najmanjše število danih količin, potrebnih za določitev koordinat geodetskih točk v izbranem koordinatnem sistemu. Z njimi definiramo lego, orientacijo in merilo geodetske mreže (Pleterski, Kregar in Urbančič, 2022). Ob morebitnih napačnih predpostavkah o stabilnosti referenčnih točk geodetske mreže se lahko pojavijo težave pri interpretaciji rezultatov. V praktičnih nalogah stremimo k temu, da referenčne točke postavimo na stabilno podlago zunaj vplivnega območja obravnavanega deformabilnega objekta in tako poskušamo zagotoviti njihovo mirovanje med izvajanjem analize. Prav tako je pomembna ustrezna geometrijska razporeditev, ki zagotavlja optimalno razporeditev pogreškov v geodetski mreži. Kontrolne točke so navadno trajno stabilizirane na preučevanem objektu. Njihove lokacije določimo v sodelovanju s strokovnjaki drugih strok. Na podlagi izračunanih rezultatov o premikih kontrolnih točk lahko ugotovimo, kaj se dogaja s preučevanim objektom, in opozorimo na morebitne nevarnosti.

V geodetski praksi so metode deformacijske analize zaradi kompleksnosti in matematičnega ozadja pogosto obravnavane kot prezahtevne, zato je metoda določitve premikov točk poenostavljena. Tako se pogosto uporablja test za ugotavljanje statistične značilnosti premika kot razmerje med premikom in pripadajočo natančnostjo premika točke. Običajno izračunano vrednost testa primerjamo s faktorjem 3,5 ali več, kar je le groba ocena (Savšek-Safić et al., 2003). V geodeziji poznamo več pristopov k deformacijski analizi: Hannover, Delft, Karlsruhe itd. (Mihailović in Aleksić, 1994). Bistvo tovrstnih pristopov je, da na podlagi več periodičnih terminskih izmer presodimo statistično značilnost premika ob predpostavkah o dejanskem tveganju za zavrnitev ničelne hipoteze in pripadajoči porazdelitveni funkciji izbrane testne statistike. Različni pristopi ne zagotovijo enolične rešitve, saj se nanašajo na različne testne statistike. V članku obravnavamo pristop *kvadratne metode* M_{split} na izbranem testnem primeru in rezultate ovrednotimo s primerjalno analizo z rezultati drugih postopkov deformacijske analize. Ker nam ni uspelo pridobiti prepričljivih rezultatov za 2D geodetsko mrežo, dobesedno sledeč postopkom Wiśniewskega (2009b, 2009c), smo morali enačbe nekoliko preurediti in uporabiti ustrezne začetne vrednosti. V tem prispevku navajamo postopek s prilagoditvami, ki smo jih morali narediti. Glavni namen članka je torej dobljene rezultate *kvadratne metode* M_{split} primerjati z rezultati drugih postopkov deformacijske analize po nekoliko preurejenih enačbah, opisanih v nadaljevanju.

Kvadratno metodo M_{split} so v deformacijski analizi uporabili že drugi avtorji. Obravnavali so predvsem 1D oziroma nivelmanske mreže (Duchnowski in Wiśniewski, 2011, 2012; Duchnowski in Wyszkowska, 2022;

Wiśniewski, 2009b, 2009c, 2010; Wiśniewski, Duchnowski in Dumalski, 2019; Wiśniewski in Zienkiewicz, 2016, 2020, 2021; Wyszkowska in Duchnowski, 2019, 2020; Zienkiewicz, 2015, 2019, 2022; Zienkiewicz in Baryla, 2015; Zienkiewicz in Dąbrowski, 2023; Zienkiewicz, Hejbudzka in Dumalski, 2017). Nekaj avtorjev je uporabilo *kvadratno metodo* M_{split} v deformacijski analizi 2D geodetskih mrež. Zienkiewicz (2019) predstavi problem robustnosti predlagane metode proti grobim pogreškom, ki se pojavljajo med meritvami. Duchnowski in Wyszkowska (2022) obravnavata nestabilne točke na objektu z deformacijsko analizo, temelječo na pristopu psevdo terminske izmere. Novel (2019) uporabi *kvadratno metodo* $M_{split(q)}$ S transformacijo v deformacijski analizi.

2 KVADRATNA METODA M_{split}

Ocena premikov točk po *kvadratni metodi* M_{split} (angl. *squared* M_{split} *estimation*) pomeni nadaljnji razvoj metode največjega verjetja. *Kvadratna metoda* M_{split} temelji na predpostavki, da lahko klasičen funkcijski model razdelimo na q konkurenčnih modelov (Wiśniewski, 2009a, 2009b, 2010). Meritve v vsakem posameznem modelu torej predstavljajo niz slučajnih spremenljivk (parametrov), ki se lahko medsebojno razlikujejo. V našem primeru obravnavanja *kvadratne metode* M_{split} privzamemo, da klasičen funkcionalni model razdelimo na dva konkurenčna funkcionalna modela. Navedeno lastnost obravnavamo tudi pri reševanju posameznih geodetskih problemov na področju robustne transformacije, deformacijske analize in robustne ocene parametrov (Wiśniewski, Duchnowski in Dumalski, 2019). Wiśniewski (2009a, 2009b, 2010) je dokazal, da je *kvadratna metoda* M_{split} alternativni pristop robustnim metodam. Pristop lahko uporabljamo tako v nivelmanskih kot v horizontalnih geodetskih mrežah.

V obravnavanem testnem primeru obravnavamo horizontalo geodetsko mrežo. Enačbe v našem prispevku so povzete po že objavljenih rezultatih raziskav (Wiśniewski, Duchnowski in Dumalski, 2019; Wiśniewski, 2009a, 2009b, 2010; Wyszkowska in Duchnowski, 2020).

Meritve in neznanke so med seboj povezane z matematični povezavami, ki so v splošnem nelinearne (npr. Ghilani, 2010, str. 189–195; Leick, 1980, str. 51–68; Leick, Rapoport in Tatarnikov, 2015, str. 17–31; Ogundare, 2019, str. 179–191):

$$\hat{\mathbf{y}} = \boldsymbol{f}(\hat{\mathbf{x}}) \text{ or } \mathbf{y} - \mathbf{v} = \boldsymbol{f}(\mathbf{x}_0 + \mathbf{x}), \quad (1)$$

kjer so:

$\hat{\mathbf{y}}$ … vektor izravnanih meritev/opazovanj,

$\hat{\mathbf{x}}$ … vektor izravnanih vrednosti neznank,

$\boldsymbol{f}(\cdot)$… nelinearne matematične funkcije,

$\mathbf{y}$ … vektor meritev/opazovanj,

$\mathbf{v}$ … vektor popravkov meritev,

$\mathbf{x}_0$ … vektor približnih vrednosti neznank,

$\mathbf{x}$ … vektor popravkov približnih vrednosti neznank.

Opozarjamo, da smo na levi strani enačbe (1) namenoma zapisali razliko, zato da se naš zapis ujema z izpeljanimi enačbami *kvadratne metode* M_{split}. V literaturi (npr. Ghilani, 2010; Leick, 1980; Leick, Rapoport in Tatarnikov, 2015; Ogundare, 2019) je na levi strani enačbe (1) vsota.

Z razvojem nelinearne enačbe (1) v Taylorjevo vrsto okrog približnih vrednosti neznank x_0 dobimo linearizirano obliko enačbe (1):

$$\mathbf{y}-\mathbf{v}=\boldsymbol{f}(\mathbf{x}_0)+\left.\frac{\partial \boldsymbol{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}_0}\mathbf{x}\ \text{oz.}\ \mathbf{f}=\mathbf{A}\mathbf{x}+\mathbf{v}, \tag{2}$$

kjer so:

$\boldsymbol{f}(\mathbf{x}_0)=\mathbf{y}_0$ … vektor vrednosti meritev, izračunanih iz približnih vrednosti neznank $\mathbf{x}_0$,

$\mathbf{A}=\left.\frac{\partial \boldsymbol{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}_0}$ … matrika koeficientov enačb popravkov ali tako imenovana matrika modela,

$\mathbf{f}=\mathbf{y}-\boldsymbol{f}(\mathbf{x}_0)=\mathbf{y}-\mathbf{y}_0$ … vektor odstopanj – neskladje meritev $\mathbf{y}$ in vrednosti merjenih količin $\mathbf{y}_0$, ki jih izračunamo na podlagi približnih vrednosti neznank $\mathbf{x}_0$.

Obravnavan pristop razdeli osnovno enačbo (2) na q delov, pri čemer je q število terenskih izmer (v virih Wiśniewski, Duchnowski in Dumalski, 2019; Wiśniewski, 2009a, 2009b, 2010; Wyszkowska in Duchnowski, 2020, je vektor meritev označen z $\mathbf{y}$, mi smo vektor meritev, glede na (2), označili s $\mathbf{f}$, kar velja do enačbe (59)):

$$\mathbf{f}=\mathbf{A}\mathbf{x}+\mathbf{v}\xrightarrow{\text{razdelimo}}\begin{cases}\mathbf{f}=\mathbf{A}\mathbf{x}_1+\mathbf{v}_1\\ \quad\vdots\\ \mathbf{f}=\mathbf{A}\mathbf{x}_q+\mathbf{v}_q\end{cases}. \tag{3}$$

Za lažjo obravnavo bodo enačbe v nadaljevanju napisane za primer, ko obravnavamo dve terenski izmeri, kar označujemo z α (1. izmera) in β (2. izmera). Enačba (3) je sedaj oblike:

$$\begin{aligned}\mathbf{f}&=\mathbf{A}\mathbf{x}_\alpha+\mathbf{v}_\alpha\\ \mathbf{f}&=\mathbf{A}\mathbf{x}_\beta+\mathbf{v}_\beta\end{aligned}. \tag{4}$$

Z obravnavnim pristopom želimo v vsaki iteraciji za vsako odstopanje v vektorju odstopanj $\mathbf{f}$ izračunati ocenjene parametre $\hat{\mathbf{x}}_\alpha$ in $\hat{\mathbf{x}}_\beta$ ter pripadajoče popravke v $\hat{\mathbf{v}}_\alpha$ in $\hat{\mathbf{v}}_\beta$, za kar obravnavamo funkcijo oblike (Wiśniewski, Duchnowski in Dumalski, 2019; Wiśniewski, 2009a, 2009b, 2010; Wyszkowska in Duchnowski, 2020):

$$\min_{\mathbf{x}_\alpha,\mathbf{x}_\beta}\varphi\left(\mathbf{f};\mathbf{x}_\alpha,\mathbf{x}_\beta\right)=\varphi\left(\mathbf{f};\hat{\mathbf{x}}_\alpha,\hat{\mathbf{x}}_\beta\right), \tag{5}$$

kjer je:

$$\aleph\left(\mathbf{f};\mathbf{x}_\alpha,\mathbf{x}_\beta\right)=\sum_{i=1}^{n}\ {}_{\alpha}\left(f_i;\mathbf{x}_\alpha\right)\ {}_{\beta}\left(f_i;\mathbf{x}_\beta\right)=\left[\boldsymbol{\rho}_\alpha\left(\mathbf{f};\mathbf{x}_\alpha\right)\right]^{\mathrm{T}}\boldsymbol{\rho}_\beta\left(\mathbf{f};\mathbf{x}_\beta\right) \tag{6}$$

in $i=1,\dots,n$, kjer je n število meritev v vseh terminskih izmerah skupaj.

Če sta funkciji $\rho_\alpha(f_i;\mathbf{x}_\alpha)$ and $\rho_\beta(f_i;\mathbf{x}_\beta)$ konveksni in za njiju obstaja odvod drugega reda, uporabimo Newtonovo metodo za rešitev problema enačbe (5) (Teunissen, 1990; Wiśniewski, 2009a, 2009b).

Parametri $\hat{\mathbf{x}}_\alpha$ in $\hat{\mathbf{x}}_\beta$ so rešitve obravnavane metode, ko velja, da je gradient funkcije (6) enak nič, torej (Wiśniewski, Duchnowski in Dumalski, 2019; Wiśniewski, 2009a, 2009b):

$$\mathbf{g}_\alpha(\mathbf{x}_\alpha,\mathbf{x}_\beta)\Big|_{\substack{\mathbf{x}_\alpha=\hat{\mathbf{x}}_\alpha\\ \mathbf{x}_\beta=\hat{\mathbf{x}}_\beta}} = \frac{\partial}{\partial \mathbf{x}_\alpha}\varphi(\mathbf{f};\mathbf{x}_\alpha,\mathbf{x}_\beta) = \mathbf{0} \text{ in} \tag{7}$$

$$\mathbf{g}_\beta(\mathbf{x}_\alpha,\mathbf{x}_\beta)\Big|_{\substack{\mathbf{x}_\alpha=\hat{\mathbf{x}}_\alpha\\ \mathbf{x}_\beta=\hat{\mathbf{x}}_\beta}} = \frac{\partial}{\partial \mathbf{x}_\beta}\varphi(\mathbf{f};\mathbf{x}_\alpha,\mathbf{x}_\beta) = \mathbf{0}. \tag{8}$$

Parcialna odvoda v enačbah (7) in (8) lahko zapišemo tudi kot (Wiśniewski, 2009a, 2010):

$$\frac{\partial}{\partial \mathbf{x}_\alpha}\varphi(\mathbf{f};\mathbf{x}_\alpha,\mathbf{x}_\beta) = \frac{\partial \mathbf{v}_\alpha}{\partial \mathbf{x}_\alpha}\frac{\partial}{\partial \mathbf{v}_\alpha}\varphi(\mathbf{f};\mathbf{x}_\alpha,\mathbf{x}_\beta) = \frac{\partial \mathbf{v}_\alpha}{\partial \mathbf{x}_\alpha}\left[\rho_\beta(v_{1\beta})\frac{\partial\rho_\alpha(v_{1\alpha})}{\partial v_{1\alpha}},\ldots,\rho_\beta(v_{n\beta})\frac{\partial\rho_\alpha(v_{n\alpha})}{\partial v_{n\alpha}}\right]^{\mathrm{T}} \text{ in} \tag{9}$$

$$\frac{\partial}{\partial \mathbf{x}_\beta}\varphi(\mathbf{f};\mathbf{x}_\alpha,\mathbf{x}_\beta) = \frac{\partial \mathbf{v}_\beta}{\partial \mathbf{x}_\beta}\frac{\partial}{\partial \mathbf{v}_\beta}\varphi(\mathbf{f};\mathbf{x}_\alpha,\mathbf{x}_\beta) = \frac{\partial \mathbf{v}_\beta}{\partial \mathbf{x}_\beta}\left[\rho_\alpha(v_{1\alpha})\frac{\partial\rho_\beta(v_{1\beta})}{\partial v_{1\beta}},\ldots,\rho_\alpha(v_{n\alpha})\frac{\partial\rho_\beta(v_{n\beta})}{\partial v_{n\beta}}\right]^{\mathrm{T}}. \tag{10}$$

V enačbah (9) in (10) lahko elemente v vektorju poenostavimo v obliko (Wiśniewski, 2009a):

$$\boldsymbol{\rho}_\alpha(\mathbf{f};\mathbf{x}_\alpha) = \left[\rho_\alpha(f_1;\mathbf{x}_\alpha),\ldots,\rho_\alpha(f_n;\mathbf{x}_\alpha)\right]^{\mathrm{T}} = \left[\rho_\alpha(v_{1\alpha}),\ldots,\rho_\alpha(v_{n\alpha})\right]^{\mathrm{T}} = \boldsymbol{\rho}_\alpha(\mathbf{v}_\alpha) \text{ in} \tag{11}$$

$$\boldsymbol{\rho}_\beta(\mathbf{f};\mathbf{x}_\beta) = \left[\rho_\beta(f_1;\mathbf{x}_\beta),\ldots,\rho_\beta(f_n;\mathbf{x}_\beta)\right]^{\mathrm{T}} = \left[\rho_\beta(v_{1\beta}),\ldots,\rho_\beta(v_{n\beta})\right]^{\mathrm{T}} = \boldsymbol{\rho}_\beta(\mathbf{v}_\beta). \tag{12}$$

Za nadaljevanje izpeljevanja rešitve pretvorimo člena $\rho_\alpha(\mathbf{v}_\alpha)$ in $\rho_\beta(\mathbf{v}_\beta)$ iz enačb (11) in (12) v diagonalno matriko (Wiśniewski, 2009a, 2010):

$$\operatorname{diag}\{\boldsymbol{\rho}_\alpha(\mathbf{v}_\alpha)\} = \operatorname{diag}\{\rho_\alpha(v_{1\alpha}),\ldots,\rho_\alpha(v_{n\alpha})\} \text{ in} \tag{13}$$

$$\operatorname{diag}\{\boldsymbol{\rho}_\beta(\mathbf{v}_\beta)\} = \operatorname{diag}\{\rho_\beta(v_{1\beta}),\ldots,\rho_\beta(v_{n\beta})\}. \tag{14}$$

Dodatno velja še (Wiśniewski, 2009a, 2010):

$$\left[\frac{\partial\rho_\alpha(v_{1\alpha})}{\partial v_{1\alpha}},\ldots,\frac{\partial\rho_\alpha(v_{n\alpha})}{\partial v_{n\alpha}}\right]^{\mathrm{T}} = \frac{\partial\boldsymbol{\rho}_\alpha(\mathbf{v}_\alpha)}{\partial\mathbf{v}_\alpha} = \mathbf{g}_{M\alpha}(\mathbf{v}_\alpha) \text{ in} \tag{15}$$

$$\left[\frac{\partial\rho_\beta(v_{1\beta})}{\partial v_{1\beta}},\ldots,\frac{\partial\rho_\beta(v_{n\beta})}{\partial v_{n\beta}}\right]^{\mathrm{T}} = \frac{\partial\boldsymbol{\rho}_\beta(\mathbf{v}_\beta)}{\partial\mathbf{v}_\beta} = \mathbf{g}_{M\beta}(\mathbf{v}_\beta), \tag{16}$$

$$\frac{\partial\mathbf{v}_\alpha}{\partial\mathbf{x}_\alpha} = \frac{\partial}{\partial\mathbf{x}_\alpha}(\mathbf{f}-\mathbf{A}\mathbf{x}_\alpha) = -\mathbf{A}^{\mathrm{T}} \text{ in} \tag{17}$$

$$\frac{\partial\mathbf{v}_\beta}{\partial\mathbf{x}_\beta} = \frac{\partial}{\partial\mathbf{x}_\beta}(\mathbf{f}-\mathbf{A}\mathbf{x}_\beta) = -\mathbf{A}^{\mathrm{T}}. \tag{18}$$

Gradienta $\mathbf{g}_\alpha(\hat{\mathbf{x}}_\alpha, \hat{\mathbf{x}}_\beta)$ in $\mathbf{g}_\beta(\hat{\mathbf{x}}_\alpha, \hat{\mathbf{x}}_\beta)$ v enačbah (7) in (8) izrazimo z upoštevanjem enačb (13)–(18) (Wiśniewski, 2009a, 2009b, 2010):

$$\mathbf{g}_\alpha(\mathbf{x}_\alpha,\mathbf{x}_\beta) = \frac{\partial}{\partial\mathbf{x}_\alpha}\varphi(\mathbf{f};\mathbf{x}_\alpha,\mathbf{x}_\beta) = -\mathbf{A}^{\mathrm{T}}\operatorname{diag}\{\boldsymbol{\rho}_\beta(\mathbf{v}_\beta)\}\,\mathbf{g}_{M\alpha}(\mathbf{v}_\alpha) \text{ in} \tag{19}$$

$$\mathbf{g}_\beta(\mathbf{x}_\alpha,\mathbf{x}_\beta) = \frac{\partial}{\partial\mathbf{x}_\beta}\varphi(\mathbf{f};\mathbf{x}_\alpha,\mathbf{x}_\beta) = -\mathbf{A}^{\mathrm{T}}\operatorname{diag}\{\boldsymbol{\rho}_\alpha(\mathbf{v}_\alpha)\}\,\mathbf{g}_{M\beta}(\mathbf{v}_\beta). \tag{20}$$

Ker obravnavamo tako imenovano *kvadratne metode* M_{split}, je enačbi (11) in (12) treba ustrezno preoblikovati (Wiśniewski, 2009b, 2010):

$$\rho_\alpha(f_i;\mathbf{x}_\alpha)=\rho_\alpha(v_{i\alpha})=v_{i\alpha}^2 \;\rightarrow\; \operatorname{diag}\{\boldsymbol{\rho}_\alpha(\mathbf{v}_\alpha)\}=\operatorname{diag}\{v_{1\alpha}^2,\ldots,v_{n\alpha}^2\}=\mathbf{w}_\beta(\mathbf{v}_\alpha) \text{ in} \tag{21}$$

$$\rho_\beta(f_i;\mathbf{x}_\beta)=\rho_\beta(v_{i\beta})=v_{i\beta}^2 \;\rightarrow\; \operatorname{diag}\{\boldsymbol{\rho}_\beta(\mathbf{v}_\beta)\}=\operatorname{diag}\{v_{1\beta}^2,\ldots,v_{n\beta}^2\}=\mathbf{w}_\alpha(\mathbf{v}_\beta), \tag{22}$$

$$\mathbf{g}_{M\alpha}(\mathbf{v}_\alpha)=2\left[v_{1\alpha},\ldots,v_{n\alpha}\right]^{\mathrm{T}}=2\mathbf{v}_\alpha \text{ in} \tag{23}$$

$$\mathbf{g}_{M\beta}(\mathbf{v}_\beta)=2\left[v_{1\beta},\ldots,v_{n\beta}\right]^{\mathrm{T}}=2\mathbf{v}_\beta. \tag{24}$$

Na podlagi izpeljav v enačbah (21)–(24) napišemo gradienta $\mathbf{g}_\alpha(\mathbf{x}_\alpha, \mathbf{x}_\beta)$ in $\mathbf{g}_\beta(\mathbf{x}_\alpha, \mathbf{x}_\beta)$ v končni obliki (Wiśniewski, 2009a, 2009b, 2010):

$$\mathbf{g}_\alpha(\mathbf{x}_\alpha,\mathbf{x}_\beta)=-\mathbf{A}^{\mathrm{T}}\operatorname{diag}\{\boldsymbol{\rho}_\beta(\mathbf{v}_\beta)\}\mathbf{g}_{M\alpha}(\mathbf{v}_\alpha)=-2\mathbf{A}^{\mathrm{T}}\mathbf{w}_\alpha(\mathbf{v}_\beta)\mathbf{v}_\alpha \text{ in} \tag{25}$$

$$\mathbf{g}_\beta(\mathbf{x}_\alpha,\mathbf{x}_\beta)=-\mathbf{A}^{\mathrm{T}}\operatorname{diag}\{\boldsymbol{\rho}_\alpha(\mathbf{v}_\alpha)\}\mathbf{g}_{M\beta}(\mathbf{v}_\beta)=-2\mathbf{A}^{\mathrm{T}}\mathbf{w}_\beta(\mathbf{v}_\alpha)\mathbf{v}_\beta. \tag{26}$$

Newtonova metoda je ena izmed iterativnih metod, s katerimi iščemo približek ničle realne funkcije. Newton je metodo reševanja nelinearne enačbe razvil iz sekantne metode in metode končnih razlik. Raphson jo je nato poenostavil in zapisal v obliki, kakršno poznamo danes, zato jo v nekaterih virih zasledimo tudi kot Newton-Raphsonovo metodo. Simpson je algoritem nekaj let kasneje prilagodil za reševanje sistema nelinearnih enačb (Močnik, 2022). Cilj metode je iz začetnega približka z iteracijskim postopkom izračunati zaporedje približkov, ki imajo za limito ničlo funkcije. Pri tem se je treba zavedati, da je izbira začetnega približka ključna, saj bo metoda ob slabem začetnem približku divergirala. V primeru dobrega približka bo konvergirala k neki ničli, pri čemer nimamo nadzora, h kateri ničli metoda konvergira. Newtonovo metodo lahko iterativno zapišemo kot (Močnik, 2022):

$$x_{j+1}=x_j-\frac{f\left(x_j\right)}{f'\left(x_j\right)},\ j=0,1,\ldots \tag{27}$$

Metodo lahko izpeljemo tudi analitično. Funkcijo f razvijemo v Taylorjevo vrsto okoli približka x_j:

$$f\left(x_j+h\right)=f\left(x_j\right)+f'\left(x_j\right)h+\frac{1}{2!}f''\left(x_j\right)h^2+\ldots \tag{28}$$

Ko preidemo iz funkcije ene spremenljivke v funkcijo več spremenljivk, linearni del Taylorjeve vrste zapišemo v obliki

$$f\left(x_j+h\right)=f\left(x_j\right)+\mathbf{J}\left(x_j\right)h, \tag{29}$$

kjer sta x_j in h vektorja dimenzije $n \times 1$, in $\mathbf{J}$ pa Jacobijeva matrika preslikave f. Newtonova metoda za večdimenzionalni primer ima torej predpis

$$x_{j+1}=x_j-\mathbf{J}^{-1}\left(x_j\right)f\left(x_j\right). \tag{30}$$

Jacobijeva matrika je matrika, sestavljena iz parcialnih odvodov prvega reda. Za rešitev problema potrebujemo Hessejevo matriko, ki je kvadratna matrika, sestavljena iz parcialnih odvodov drugega reda. Velja, da je Jacobijeva matrika gradienta funkcije f, označimo jo z ∇f, enaka Hessejevi matriki. To zapišemo kot:

$$\mathbf{H}(x)=\mathbf{J}(\nabla f). \tag{31}$$

Pri *kvadratni metodi* M_{split}, Jacobijevo matriko predstavlja gradient funkcij (7) in (8). Hessejevo matriko tako dobimo ob dvakratnem odvajanju funkcije (6) oziroma odvajanju gradienta funkcij (7) in (8). Pri tem velja (Wiśniewski, 2009a, 2009b, 2010):

$$\mathbf{H}_\alpha(\mathbf{x}_\alpha,\mathbf{x}_\beta)=\frac{\partial^2}{\partial\mathbf{x}_\alpha\partial\mathbf{x}_\alpha^T}\varphi(\mathbf{f};\mathbf{x}_\alpha,\mathbf{x}_\beta)=\frac{\partial}{\partial\mathbf{x}_\alpha^T}\mathbf{g}_\alpha(\mathbf{x}_\alpha,\mathbf{x}_\beta)\text{ in} \quad (32)$$

$$\mathbf{H}_\beta(\mathbf{x}_\alpha,\mathbf{x}_\beta)=\frac{\partial^2}{\partial\mathbf{x}_\beta\partial\mathbf{x}_\beta^T}\varphi(\mathbf{f};\mathbf{x}_\alpha,\mathbf{x}_\beta)=\frac{\partial}{\partial\mathbf{x}_\beta^T}\mathbf{g}_\beta(\mathbf{x}_\alpha,\mathbf{x}_\beta). \quad (33)$$

Iz enačb (19) in (20) sledi (Wiśniewski, 2009a, 2009b, 2010):

$$\frac{\partial}{\partial\mathbf{x}_\alpha^T}\mathbf{g}_\alpha(\mathbf{x}_\alpha,\mathbf{x}_\beta)=-\mathbf{A}^T\mathrm{diag}\left\{\boldsymbol{\rho}_\beta(\mathbf{v}_\beta)\right\}\frac{\partial\mathbf{g}_{M\alpha}(\mathbf{v}_\alpha)}{\partial\mathbf{v}_\alpha^T}\frac{\partial\mathbf{v}_\alpha}{\partial\mathbf{x}_\alpha^T}\text{ in} \quad (34)$$

$$\frac{\partial}{\partial\mathbf{x}_\beta^T}\mathbf{g}_\beta(\mathbf{x}_\alpha,\mathbf{x}_\beta)=-\mathbf{A}^T\mathrm{diag}\left\{\boldsymbol{\rho}_\alpha(\mathbf{v}_\alpha)\right\}\frac{\partial\mathbf{g}_{M\beta}(\mathbf{v}_\beta)}{\partial\mathbf{v}_\beta^T}\frac{\partial\mathbf{v}_\beta}{\partial\mathbf{x}_\beta^T}, \quad (35)$$

kjer je:

$$\frac{\partial\mathbf{g}_{M\alpha}(\mathbf{v}_\alpha)}{\partial\mathbf{v}_\alpha^T}=\mathbf{H}_{M\alpha}(\mathbf{v}_\alpha)\text{ in} \quad (36)$$

$$\frac{\partial\mathbf{g}_{M\beta}(\mathbf{v}_\beta)}{\partial\mathbf{v}_\beta^T}=\mathbf{H}_{M\beta}(\mathbf{v}_\beta), \quad (37)$$

$$\frac{\partial\mathbf{v}_\alpha}{\partial\mathbf{x}_\alpha^T}=\frac{\partial}{\partial\mathbf{x}_\alpha^T}\left(\mathbf{f}-\mathbf{A}\mathbf{x}_\alpha\right)=-\mathbf{A}\text{ in} \quad (38)$$

$$\frac{\partial\mathbf{v}_\beta}{\partial\mathbf{x}_\beta^T}=\frac{\partial}{\partial\mathbf{x}_\beta^T}\left(\mathbf{f}-\mathbf{A}\mathbf{x}_\beta\right)=-\mathbf{A}. \quad (39)$$

Glede na enačbe (32)–(39) preoblikujemo Hessejevo matriko v (Wiśniewski, 2009a, 2009b):

$$\mathbf{H}_\alpha(\mathbf{x}_\alpha,\mathbf{x}_\beta)=-\mathbf{A}^T\mathrm{diag}\left\{\boldsymbol{\rho}_\beta(\mathbf{v}_\beta)\right\}\frac{\partial\mathbf{g}_{M\alpha}(\mathbf{v}_\alpha)}{\partial\mathbf{v}_\alpha^T}\frac{\partial\mathbf{v}_\alpha}{\partial\mathbf{x}_\alpha^T}=\mathbf{A}^T\mathrm{diag}\left\{\boldsymbol{\rho}_\beta(\mathbf{v}_\beta)\right\}\mathbf{H}_{M\alpha}(\mathbf{v}_\alpha)\mathbf{A}\text{ in} \quad (40)$$

$$\mathbf{H}_\beta(\mathbf{x}_\alpha,\mathbf{x}_\beta)=-\mathbf{A}^T\mathrm{diag}\left\{\boldsymbol{\rho}_\alpha(\mathbf{v}_\alpha)\right\}\frac{\partial\mathbf{g}_{M\beta}(\mathbf{v}_\beta)}{\partial\mathbf{v}_\beta^T}\frac{\partial\mathbf{v}_\beta}{\partial\mathbf{x}_\beta^T}=\mathbf{A}^T\mathrm{diag}\left\{\boldsymbol{\rho}_\alpha(\mathbf{v}_\alpha)\right\}\mathbf{H}_{M\beta}(\mathbf{v}_\beta)\mathbf{A}. \quad (41)$$

Iz enačb (23) in (24) lahko ugotovimo, da velja (Wiśniewski, 2009a, 2009b):

$$\mathbf{H}_{M\alpha}(\mathbf{v}_\alpha)=\frac{\partial\mathbf{g}_{M\alpha}(\mathbf{v}_\alpha)}{\partial\mathbf{v}_\alpha^T}=\frac{\partial(2\mathbf{v}_\alpha)}{\partial\mathbf{v}_\alpha^T}=2\mathbf{I}\text{ in} \quad (42)$$

$$\mathbf{H}_{M\beta}(\mathbf{v}_\beta)=\frac{\partial\mathbf{g}_{M\beta}(\mathbf{v}_\beta)}{\partial\mathbf{v}_\beta^T}=\frac{\partial(2\mathbf{v}_\beta)}{\partial\mathbf{v}_\beta^T}=2\mathbf{I}, \quad (43)$$

kjer je **I** enotska matrika.

Hessejevo matriko v končni obliki zapišemo (Wiśniewski, 2009a, 2009b, 2010):

$$\mathbf{H}_\alpha(\mathbf{x}_\alpha,\mathbf{x}_\beta)=\mathbf{A}^T\mathrm{diag}\left\{\boldsymbol{\rho}_\beta(\mathbf{v}_\beta)\right\}\mathbf{H}_{M\alpha}(\mathbf{v}_\alpha)\mathbf{A}=2\mathbf{A}^T\mathbf{w}_\alpha(\mathbf{v}_\beta)\mathbf{A}=\mathbf{H}_\alpha(\mathbf{x}_\beta)\text{ in} \quad (44)$$

$$\mathbf{H}_\beta(\mathbf{x}_\alpha,\mathbf{x}_\beta)=\mathbf{A}^T\mathrm{diag}\left\{\boldsymbol{\rho}_\alpha(\mathbf{v}_\alpha)\right\}\mathbf{H}_{M\beta}(\mathbf{v}_\beta)\mathbf{A}=2\mathbf{A}^T\mathbf{w}_\beta(\mathbf{v}_\alpha)\mathbf{A}=\mathbf{H}_\beta(\mathbf{x}_\alpha). \quad (45)$$

Na podlagi do sedaj predstavljenih enačb in izpeljav ter upoštevaje enačbo (30) preidemo do končne rešitve *kvadratne metode* M_{split} oziroma do iterativnega postopka računanja rešitve (Wiśniewski, 2009a, 2009b, 2010; Wyszkowska in Duchnowski, 2020):

$$\mathbf{x}_\alpha^j = \mathbf{x}_\alpha^{j-1} + \Delta\mathbf{x}_\alpha^j;\ j = 1,\ldots,k \text{ in} \tag{46}$$

$$\mathbf{x}_\beta^j = \mathbf{x}_\beta^{j-1} + \Delta\mathbf{x}_\beta^j;\ j = 1,\ldots,k, \tag{47}$$

kjer je:

k … število iteracij,

$$\begin{aligned}\Delta\mathbf{x}_\alpha^j &= -\left\{\mathbf{H}_\alpha\left(\mathbf{x}_\alpha^{j-1},\mathbf{x}_\beta^{j-1}\right)\right\}^{-1}\mathbf{g}_\alpha\left(\mathbf{x}_\alpha^{j-1},\mathbf{x}_\beta^{j-1}\right)\\ &= -\left\{\mathbf{H}_\alpha\left(\mathbf{x}_\beta^{j-1}\right)\right\}^{-1}\mathbf{g}_\alpha\left(\mathbf{x}_\alpha^{j-1},\mathbf{x}_\beta^{j-1}\right)\\ &= \left\{\mathbf{A}^{\mathrm{T}}\mathrm{diag}\left\{\boldsymbol{\rho}_\beta\left(\mathbf{v}_\beta^{j-1}\right)\right\}\mathbf{H}_{M\alpha}\left(\mathbf{v}_\alpha^{j-1}\right)\mathbf{A}\right\}^{-1}\mathbf{A}^{\mathrm{T}}\mathrm{diag}\left\{\boldsymbol{\rho}_\beta\left(\mathbf{v}_\beta^{j-1}\right)\right\}\mathbf{g}_{M\alpha}\left(\mathbf{v}_\alpha^{j-1}\right)\\ &= \left\{\mathbf{A}^{\mathrm{T}}\mathbf{w}_\alpha\left(\mathbf{v}_\beta^{j-1}\right)2\mathbf{A}\right\}^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{w}_\alpha\left(\mathbf{v}_\beta^{j-1}\right)2\mathbf{v}_\alpha^{j-1}\\ &= \left\{\mathbf{A}^{\mathrm{T}}\mathbf{w}_\alpha\left(\mathbf{v}_\beta^{j-1}\right)\mathbf{A}\right\}^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{w}_\alpha\left(\mathbf{v}_\beta^{j-1}\right)\mathbf{v}_\alpha^{j-1},\end{aligned} \tag{48}$$

$$\mathbf{w}_\alpha\left(\mathbf{v}_\beta^{j-1}\right) = \mathrm{diag}\left\{\left(v_{1\beta}^{j-1}\right)^2,\ldots,\left(v_{n\beta}^{j-1}\right)^2\right\}, \tag{49}$$

$$\mathbf{v}_\alpha^j = \mathbf{f} - \mathbf{A}\mathbf{x}_\alpha^j \text{ in} \tag{50}$$

$$\begin{aligned}\Delta\mathbf{x}_\beta^j &= -\left\{\mathbf{H}_\beta\left(\mathbf{x}_\alpha^{j},\mathbf{x}_\beta^{j-1}\right)\right\}^{-1}\mathbf{g}_\beta\left(\mathbf{x}_\alpha^{j},\mathbf{x}_\beta^{j-1}\right)\\ &= -\left\{\mathbf{H}_\beta\left(\mathbf{x}_\alpha^{j}\right)\right\}^{-1}\mathbf{g}_\beta\left(\mathbf{x}_\alpha^{j},\mathbf{x}_\beta^{j-1}\right)\\ &= \left\{\mathbf{A}^{\mathrm{T}}\mathrm{diag}\left\{\boldsymbol{\rho}_\alpha\left(\mathbf{v}_\alpha^{j}\right)\right\}\mathbf{H}_{M\beta}\left(\mathbf{v}_\beta^{j-1}\right)\mathbf{A}\right\}^{-1}\mathbf{A}^{\mathrm{T}}\mathrm{diag}\left\{\boldsymbol{\rho}_\alpha\left(\mathbf{v}_\alpha^{j}\right)\right\}\mathbf{g}_{M\beta}\left(\mathbf{v}_\beta^{j-1}\right)\\ &= \left\{\mathbf{A}^{\mathrm{T}}\mathbf{w}_\beta\left(\mathbf{v}_\alpha^{j}\right)2\mathbf{A}\right\}^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{w}_\beta\left(\mathbf{v}_\alpha^{j}\right)2\mathbf{v}_\beta^{j-1}\\ &= \left\{\mathbf{A}^{\mathrm{T}}\mathbf{w}_\beta\left(\mathbf{v}_\alpha^{j}\right)\mathbf{A}\right\}^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{w}_\beta\left(\mathbf{v}_\alpha^{j}\right)\mathbf{v}_\beta^{j-1},\end{aligned} \tag{51}$$

$$\mathbf{w}_\beta\left(\mathbf{v}_\alpha^{j}\right) = \mathrm{diag}\left\{\left(v_{1\alpha}^{j}\right)^2,\ldots,\left(v_{n\alpha}^{j}\right)^2\right\}, \tag{52}$$

$$\mathbf{v}_\beta^j = \mathbf{f} - \mathbf{A}\mathbf{x}_\beta^j. \tag{53}$$

Kvadratna metoda M_{split} je iterativen postopek reševanja optimizacijskega problema. Za začetne vrednosti izračuna prametrov $\mathbf{x}_\alpha^{\mathbf{0}}$ in $\mathbf{v}_\alpha^{\mathbf{0}}$ izberemo rezultate, izračunane po metodi najmanjših kvadratov (MNK):

$$\hat{\mathbf{x}}_{\mathrm{MNK}} = \left(\mathbf{A}^{\mathbf{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{f}, \tag{54}$$

$$\hat{\mathbf{v}}_{\mathrm{MNK}} = \mathbf{f} - \mathbf{A}^{\mathbf{T}}\hat{\mathbf{x}}_{\mathrm{MNK}}. \tag{55}$$

V začetnem koraku torej privzamemo (Wiśniewski, 2009b, 2010)

$$\mathbf{x}_\alpha^0 = \hat{\mathbf{x}}_{\mathrm{MNK}} \text{ in} \tag{56}$$

$$\mathbf{v}_\alpha^0 = \hat{\mathbf{v}}_{\mathrm{MNK}}, \tag{57}$$

in izračunamo

$$\mathbf{x}_\beta^0 = \hat{\mathbf{x}}_{\text{MNK}} + \left\{ \mathbf{A}^{\text{T}} \mathbf{w}_\beta \left(\hat{\mathbf{v}}_{\text{MNK}} \right) \mathbf{A} \right\}^{-1} \mathbf{A}^{\text{T}} \mathbf{w}_\beta \left(\hat{\mathbf{v}}_{\text{MNK}} \right) \mathbf{f} \text{ in} \tag{58}$$

$$\mathbf{v}_\beta^0 = \mathbf{f} - \mathbf{A}\mathbf{x}_\beta^0, \tag{59}$$

kjer je:

$$\mathbf{w}_\beta \left(\hat{\mathbf{v}}_{\text{MNK}} \right) = \text{diag}\left\{ \hat{v}_{1\text{MNK}}^2, \ldots, \hat{v}_{n\text{MNK}}^2 \right\}. \tag{60}$$

Vse nadaljnje korake iteracije nato računamo po enačbah (46) oziroma (47). Ob predpostavki konvergence smo za kriterij ustavitve iteracijskega postopka izbrali normi vektorjev popravkov približnih koordinat (Wyszkowska in Duchnowski, 2020):

$$\left\| \Delta\mathbf{x}_\alpha^j \right\| < \varepsilon \text{ in} \tag{61}$$

$$\left\| \Delta\mathbf{x}_\beta^j \right\| < \varepsilon, \tag{62}$$

kjer je ε izbrana meja za končanje iteracijskega postopka.

$\hat{\mathbf{x}}_\alpha^k$ and $\hat{\mathbf{x}}_\beta^k$, ki ju dobimo v zadnji iteraciji *k*, sta torej končna rezultata *kvadratne metode* M_{split}. Na koncu lahko izračunamo premik posamezne točke v obravnavani geodetski mreži kot:

$$\mathbf{p} = \left(\hat{\mathbf{x}}_\beta^k - \hat{\mathbf{x}}_\alpha^k \right). \tag{63}$$

Težava tovrstnega postopka v primerjavi z vsemi drugimi zgoraj naštetimi postopki deformacijske analize je, da ne vključuje statističnega testa, na podlagi katerega bi lahko ugotovili, ali je premik statistično značilen ali ne. Rezultat postopka je le velikost premika, interpretacija rezultata pa je na strani geodeta.

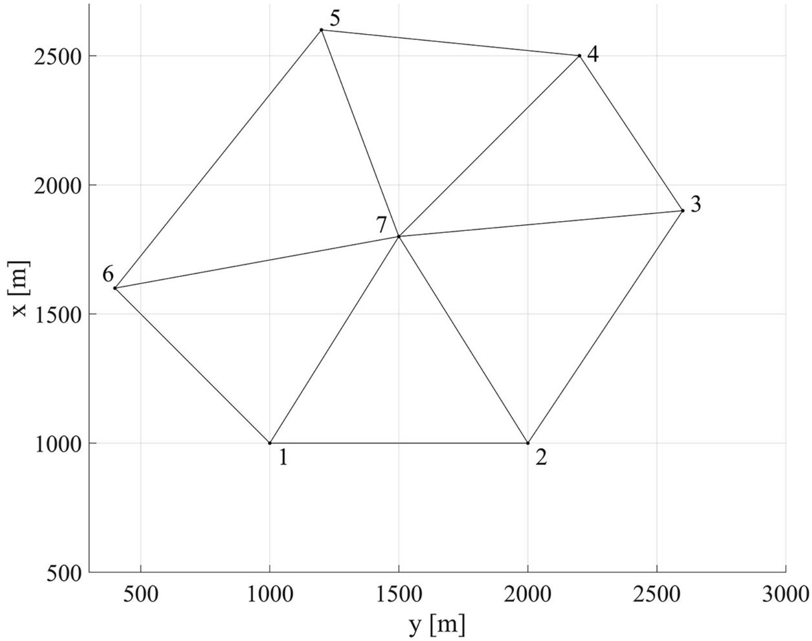
3 RAČUNSKI PRIMER

Uporabnost *kvadratne metode* M_{split} želimo prikazati na primeru iz literature (Mihailović in Aleksić, 1994).

Simulirana geodetska mreža na sliki 1 je sestavljena iz 7 točk ter merjenih 24 horizontalnih smeri in 24 dolžin. Za vrednost a-priori variance za smeri izberemo $\sigma_{Hz} = 1''$, za vrednost a-priori variance za dolžine pa $\sigma_d = 5$ mm. Za obe terminski izmeri α (1. izmera) in β (2. izmera) je oblika mreže enaka. Isti primer relativne geodetske mreže je bil obdelan tudi z drugimi postopki deformacijske analize:

- Hannover (postopek je razvil H. Pelzer – Pelzer, 1971; Ambrožič, 2001),
- Karlsruhe (postopek so razvili K. R. Koch, B. Heck, E. Kuntz in B. Meier-Hirmer – Heck, Kuntz in Meier-Hirmer, 1977; Ambrožič, 2004),
- Delft (postopek sta razvila J. van Mierlo in J. J. Kok – Heck et al., 1982; Marjetič, Zemljak in Ambrožič, 2013),
- Fredericton (postopek so razvili A. Chrzanowski, Y.-Q. Chen in J. M Secord – Chen, Chrzanowski in Secord, 1990; Vrečko in Ambrožič, 2013),
- München (postopek je razvil W. Welsch – Welsch, 1982; Soldo in Ambrožič, 2018),
- robustne metode (postopek iterativnega prilagajanja uteži je predstavil Y.-Q. Chen – Chen, 1983; Ambrožič et al., 2019),

- deformacijska analiza po postopku Caspary (postopek je razvil W. F. Caspary – Caspary, 2000; Hamza, Stopar in Ambrožič, 2020).



Slika 1: Skica mreže.

Glede na izpeljane enačbe *kvadratne metode* M_{split} v 2. poglavju so vhodni podatki za izračun matrika koeficientov enačb popravkov A in vektor odstopanj f, glej enačbo (2).

Ker obravnavamo dve terenski izmeri, morata matrika **A** in vektor **f** vsebovati elemente, ki se nanašajo na primer na smeri in nato dolžine prve terminske izmere, potem pa še elemente, ki se nanašajo na smeri in nato dolžine druge terminske izmere. Elemente matrike **A** in vektorja **f** za merjeno smer izračunamo na primer po enačbah 7.51 in 7.52 (Mihailović, 1981, str. 313):

$$v_{ri} = a_{ri} x_r + b_{ri} y_r + a_{ir} x_i + b_{ir} y_i + z_r + f_{ri}, \quad (64)$$

$a_{ri} = -\frac{\sin n_{ri}}{s^0_{ri}}, b_{ri} = \frac{\cos n_{ri}}{s^0_{ri}}, a_{ir} = -a_{ri}, b_{ir} = -b_{ri} \ldots$ elementi matrike **A**,

x_r, y_r, x_i, y_i in z_r … popravki približnih vrednosti koordinatnih neznank točk *r* in *i* ter popravek orientacijskega kota na točki *r*,

$f_{ri} = n_{ri} + z^0_r - \alpha_{ri}$ … odstopanje,

n_{ri} … približni smerni kot od točke *r* proti *i*,

z^0_r … približni orientacijski kot na točki *r*,

α_{ri} … merjena smer od točke *r* proti *i*,

S^0_{ri} … približna dolžina med točko *r* in *i*, izračunana iz približnih koordinat točk *r* in *i*.

Elemente matrike **A** in vektorja **f** za merjeno dolžino, izračunamo po enačbah 8.37 (Mihailović, 1981, str. 408):

$$v_{ri} = a_{ri} x_r + b_{ri} y_r + a_{ir} x_i + b_{ir} y_i + f_{ri}, \tag{65}$$

$a_{ri} = \cos n_{ri},\ b_{ri} = \sin n_{ri},\ a_{ir} = -a_{ri},\ b_{ir} = -b_{ri}$ … elementi matrike **A**,

$f_{ri} = S^0_{ri} - S_{ri}$ … odstopanje,

S_{ri}… merjena dolžina med točko r in i.

Seveda moramo elemente, ki se nanašajo na orientacijske neznanke, eliminirati iz sistema (2), na primer z Gaußovo eliminacijo. Ker imamo v 2D geodetskih mrežah opraviti z različnimi vrstami meritev (smeri in dolžine), ki so v splošnem različne natančnosti, moramo v enačbah *kvadratne metode* M_{split} upoštevati uteži (Zienkiewicz, Hejbudzka in Dumalski, 2017; Zienkiewicz in Baryla, 2015) ali uporabiti ekvivalentne enačbe (2) oziroma homogenizirati matriko koeficientov enačb popravkov **A** in vektor odstopanj **f**, na primer z uporabo tretjega Schreiberjevega pravila.

Popravke približnih vrednosti koordinatnih neznank $\mathbf{x}^0_\alpha$ v začetni iteraciji postopka izračuna parametrov $\hat{\mathbf{x}}_\alpha$ in $\hat{\mathbf{x}}_\beta$ *kvadratne metode* M_{split} izračunamo z metodo najmanjših kvadratov $\hat{\mathbf{x}}_{MNK}$ po enačbi (56) in jih podajamo v preglednici 1. Opozarjamo, da te vrednosti niso enake navedenim v preglednici 2 v Ambrožič (2001), saj pravkar izračunane vrednosti dobimo iz izravnave obeh terminskih izmer hkrati, vrednosti, navedene v preglednici 2 v Ambrožič (2001), pa smo dobili z izravnavo vsake terminske izmere posebej. Po enačbi (58) izračunamo še popravke približnih vrednosti koordinatnih neznank $\mathbf{x}^0_\beta$, ki jih podajamo v preglednici 1.

Preglednica 1: Popravki približnih vrednosti koordinatnih neznank $\mathbf{x}^0_\alpha$ in $\mathbf{x}^0_\beta$ [m] v začetni iteraciji

Točka *i*	$\mathbf{x}^0_\alpha = \hat{\mathbf{x}}_{MNK}$ po (56)		$\mathbf{x}^0_\beta$ po (58)	
	$\mathbf{x}^0_\alpha$ za y_i	$\mathbf{x}^0_\alpha$ za x_i	$\mathbf{x}^0_\beta$ za y_i	$\mathbf{x}^0_\beta$ za x_i
1	–0,0083	–0,0216	–0,0182	–0,0417
2	–0,0142	0,0267	–0,0311	0,0553
3	0,0143	–0,0193	0,0305	–0,0379
4	–0,0004	0,0045	0,0012	0,0098
5	–0,0048	–0,0047	–0,0104	–0,0123
6	–0,0009	–0,0073	–0,0024	–0,0165
7	0,0143	0,0218	0,0304	0,0435

V začetni iteraciji izračunamo popravke meritev $\mathbf{v}^0_\alpha = \hat{\mathbf{v}}_{LSM}$, enačba (57), in $\mathbf{v}^0_\beta = \mathbf{f} - \mathbf{A}\mathbf{x}^0_\beta$, enačba (59).

Izračun prametrov $\hat{\mathbf{x}}_\alpha$ in $\hat{\mathbf{x}}_\beta$ *kvadratne metode* M_{split} nadaljujemo z iteracijskim postopkom po enačbah od (46) do (53). Rezultate $\mathbf{x}^j_\alpha$ podajamo v preglednici 2, rezultate $\mathbf{x}^j_\beta$ pa v preglednici 3. Iteracijski postopek končamo, ko sta izpolnjena pogoja (59) in (60). Mejo za končanje iteracijskega postopka izberemo $\varepsilon = 0{,}001$. Pogoja za končanje iteracijskega postopek sta bila izpolnjena po 8. iteraciji.

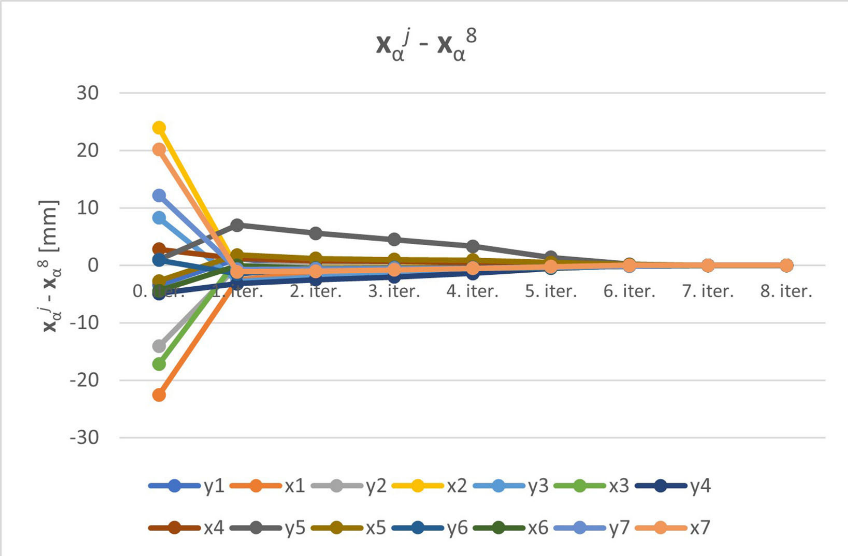
Preglednica 2: Popravki približnih vrednosti koordinatnih neznank $\mathbf{x}_\alpha^j$ [mm]

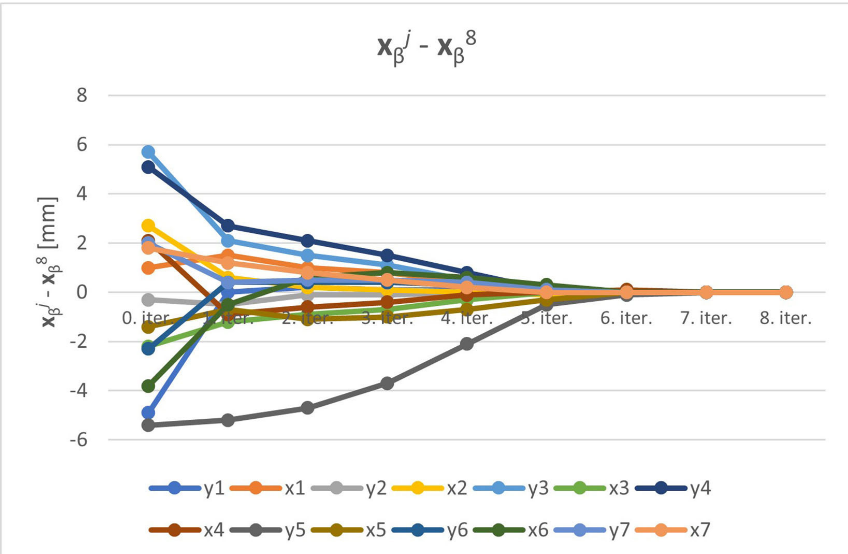
Koordinata	$\mathbf{x}_\alpha^0 = \hat{\mathbf{x}}_{LSM}$	$\mathbf{x}_\alpha^1$	$\mathbf{x}_\alpha^2$	$\mathbf{x}_\alpha^3$	$\mathbf{x}_\alpha^4$	$\mathbf{x}_\alpha^5$	$\mathbf{x}_\alpha^6$	$\mathbf{x}_\alpha^7$	$\mathbf{x}_\alpha^8$
y_1	−8,3	−3,7	−5,0	−5,0	−5,0	−4,9	−4,9	−4,9	−4,9
x_1	−21,6	−1,5	−0,3	0,0	0,3	0,7	1,0	1,0	1,0
y_2	−14,2	−0,2	0,1	0,1	−0,0	−0,1	−0,1	−0,1	−0,1
x_2	26,7	2,1	2,3	2,4	2,6	2,7	2,7	2,7	2,7
y_3	14,3	3,2	4,0	4,4	4,9	5,6	5,9	6,0	6,0
x_3	−19,3	−1,0	−1,0	−1,2	−1,5	−1,9	−2,1	−2,1	−2,1
y_4	−0,4	1,3	2,0	2,5	3,1	4,0	4,5	4,5	4,5
x_4	4,5	2,9	2,5	2,4	2,1	1,8	1,7	1,7	1,7
y_5	−4,8	1,2	−0,2	−1,3	−2,5	−4,4	−5,6	−5,8	−5,8
x_5	−4,7	−0,2	−0,8	−1,0	−1,1	−1,5	−1,9	−2,0	−2,0
y_6	−0,9	−3,3	−2,3	−2,2	−2,2	−2,0	−1,9	−1,8	−1,8
x_6	−7,3	−2,9	−3,4	−3,4	−3,4	−3,3	−3,0	−2,9	−2,9
y_7	14,3	1,4	1,5	1,6	1,6	1,8	2,0	2,1	2,1
x_7	21,8	0,5	0,6	0,8	1,1	1,4	1,6	1,6	1,6

Preglednica 3: Popravki približnih vrednosti koordinatnih neznank $\mathbf{x}_\beta^j$ [mm]

Koordinata	$\mathbf{x}_\beta^0$	$\mathbf{x}_\beta^1$	$\mathbf{x}_\beta^2$	$\mathbf{x}_\beta^3$	$\mathbf{x}_\beta^4$	$\mathbf{x}_\beta^5$	$\mathbf{x}_\beta^6$	$\mathbf{x}_\beta^7$	$\mathbf{x}_\beta^8$
y_1	−18,2	−13,3	−13,1	−13,2	−13,2	−13,3	−13,3	−13,3	−13,3
x_1	−41,7	−41,2	−41,7	−41,9	−42,3	−42,6	−42,7	−42,7	−42,7
y_2	−31,1	−31,3	−30,9	−30,9	−30,8	−30,8	−30,8	−30,8	−30,8
x_2	55,3	53,2	52,8	52,7	52,6	52,5	52,6	52,6	52,6
y_3	30,5	26,9	26,3	25,9	25,3	24,8	24,8	24,8	24,8
x_3	−37,9	−36,9	−36,6	−36,4	−36,0	−35,7	−35,7	−35,7	−35,7
y_4	1,2	−1,2	−1,8	−2,4	−3,1	−3,8	−3,9	−3,9	−3,9
x_4	9,8	6,8	7,1	7,3	7,6	7,7	7,8	7,7	7,7
y_5	−10,4	−10,2	−9,7	−8,7	−7,1	−5,5	−5,1	−5,0	−5,0
x_5	−12,3	−11,6	−12,0	−11,9	−11,6	−11,2	−10,9	−10,9	−10,9
y_6	−2,4	0,3	0,3	0,3	0,2	−0,0	−0,1	−0,1	−0,1
x_6	−16,5	−13,2	−12,1	−11,9	−12,1	−12,4	−12,7	−12,7	−12,7
y_7	30,4	28,8	28,9	28,9	28,8	28,5	28,4	28,4	28,4
x_7	43,5	42,9	42,5	42,2	41,9	41,7	41,7	41,7	41,7

Iz preglednic 2 in 3 vidimo, da so se popravki približnih vrednosti koordinatnih neznank najbolj spremenile v 1. iteraciji. Gradienti za posamezne koordinate so se torej najbolj spremenili v 1. iteraciji in so se bližali vrednosti nič v osmi iteraciji. Točno nič niso dosegli zaradi pogojev (61) in (62). Spremembe popravkov približnih vrednosti koordinatnih neznank nazorno prikazujemo še na sliki 2, kjer smo vrednostim $\mathbf{x}_\alpha^j$ odšteli $\mathbf{x}_\alpha^8$ in vrednostim $\mathbf{x}_\beta^j$ odšteli $\mathbf{x}_\beta^8$.





Slika 2: Spremembe popravkov približnih vrednosti koordinatnih neznank v iteracijah.

Na koncu izračunamo premik posamezne točke v obravnavani 2D geodetski mreži po enačbi (61), rezultate prikazujemo v preglednici 4.

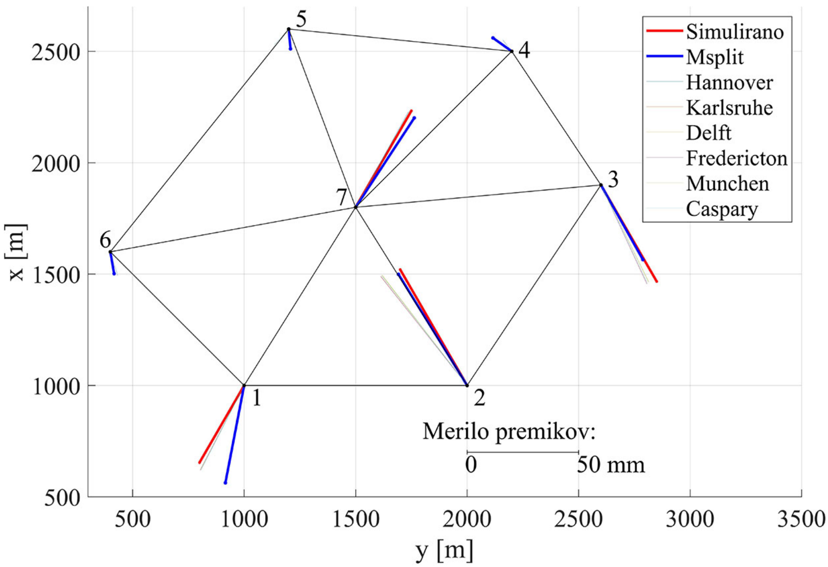
4 PRIMERJAVA Z REZULTATI DRUGIH POSTOPKOV

V preglednici 4 prikazujemo primerjavo rezultatov postopka *kvadratne metode* M_{split} in rezultatov postopkov Hannover, Karlsruhe, Delft, Fredericton, München in Caspary. V primerjavi z naštetimi postopki opazimo manjše razlike v rezultatih izračunanih premikov, kar je pričakovano.

Preglednica 4: Simulirani premiki točk mreže in rezultati deformacijske analize po postopkih Hannover, Karlsruhe, Delft, Fredericton, München, Caspary in *Kvadratne metode* M_{split}

Točka		**1**	**2**	**3**	**4**	**5**	**6**	**7**
Simulirano	d_y [mm]	–20,0	–30,0	25,0	0,0	0,0	0,0	25,0
	d_x [mm]	–34,6	52,0	–43,3	0,0	0,0	0,0	43,3
	d [mm]	40,0	60,0	50,0	0,0	0,0	0,0	50,0
	υ [°]	210	330	150	–	–	–	30
Hannover	d_y [mm]	–19,6	–38,7	20,6	–4,0	–6,4	3,3	23,6
	d_x [mm]	–38,0	49,0	–44,3	5,1	–7,1	–10,6	42,9
	d [mm]	42,8	62,4	48,9	6,5	10,0	11,1	49,0
	υ [°]	207	322	155	322	222	163	29
	Premik	da	da	da	ne	ne	ne	da
Karlsruhe	d_y [mm]	–19,7	–38,8	20,6	–	–	–	23,6
	d_x [mm]	–38,0	49,0	–44,4	–	–	–	42,9
	d [mm]	42,8	62,5	48,9	–	–	–	49,0
	υ [°]	207	322	155	–	–	–	29
	Premik	da	da	da	ne	ne	ne	da
Delft	d_y [mm]	–19,4	–38,1	21,4	0,7	–0,8	0,0	24,0
	d_x [mm]	–37,5	49,5	–43,5	1,0	–2,3	1,3	42,9
	d [mm]	42,2	62,5	48,5	1,2	2,4	1,3	49,2
	υ [°]	207	322	154	35	199	0	29
	Premik	da	da	da	ne	ne	ne	da
Fredericton	d_y [mm]	–19,6	–38,7	20,6	–	–	–	23,6
	d_x [mm]	–38,0	49,0	–44,3	–	–	–	42,9
	d [mm]	42,8	62,5	48,9	–	–	–	48,9
	υ [°]	207	322	155	–	–	–	29
	Premik	da	da	da	ne	ne	ne	da
München	d_y [mm]	–19,5	–38,2	21,4	0,7	–0,8	0,0	24,0
	d_x [mm]	–37,6	49,5	–43,6	1,0	–2,2	1,4	42,9
	d [mm]	42,4	62,5	48,6	1,2	2,3	1,4	49,2
	υ [°]	207	322	154	35	200	0	29
	Premik	da	da	da	ne	ne	ne	da
Caspary	d_y [mm]	–19,2	–38,4	20,8	–	–	–	23,9
	d_x [mm]	–37,9	49,4	–43,9	–	–	–	43,1
	d [mm]	42,5	62,5	48,6	–	–	–	49,2
	υ [°]	207	322	154	–	–	–	9
	Premik	da	da	da	ne	ne	ne	da
$\mathbf{M_{split}}$	d_y [mm]	**–8,4**	**–30,8**	**18,8**	**–8,4**	**0,8**	**1,7**	**26,3**
	d_x [mm]	**–43,7**	**49,9**	**–33,5**	**6,0**	**–8,9**	**–9,8**	**40,1**
	d [mm]	**44,5**	**58,6**	**38,5**	**10,4**	**9,0**	**10,0**	**48,0**
	υ [°]	**191**	**328**	**151**	**306**	**175**	**170**	**33**
	Premik	–	–	–	–	–	–	–

V preglednici 4 smo uporabili take oznake kot v drugih člankih, kjer smo obravnavali druge metode deformacijske analize. Tako oznaka d_y pomeni $\mathbf{p}_y = (\hat{\mathbf{x}}_\beta^k - \hat{\mathbf{x}}_\alpha^k)_y$, enačba (63), torej razlika popravkov približnih vrednosti koordinatnih neznank za posamezno točko, izračunanih v zadnji iteraciji k, oznaka d_x pomeni $\mathbf{p}_x = (\hat{\mathbf{x}}_\beta^k - \hat{\mathbf{x}}_\alpha^k)_x$, oznaka d je premik posamezne točke $\breve{}\ = \sqrt{{}_y^2 + {}_x^2}$, oznaka υ je smerni kot premika posamezne točke $\upsilon = \mathrm{atan}(d_y / d_x)$, oznaka *Premik* označuje, ali se je točka premaknila ali ne.



Slika 3: Izris premikov, izračunanih z različnimi metodami deformacijske analize.

Iz preglednice 4 in slike 3 vidimo, da so razlike v položajih točk od simuliranih vrednosti kar na štirih (od sedmih) točkah mreže največje ravno pri tej metodi, kar nas navaja k sklepu, da je metoda zgolj pogojno uporabna. Na točki 3 doseže razlika položaja po tej metodi celo največjo vrednost med vsemi obravnavanimi metodami (11,5 mm). Po drugi strani pa smo s *kvadratno metodo* M_{split} dobili popolnoma enako razliko kot z drugimi metodami, da so se točke 1, 2, 3 in 7 znatno premaknile, točke 4, 5 in 6 pa malenkostno, kar je za deformacijsko analizo pomembno.

5 ZAKLJUČEK

V tem prispevku opisujemo in razpravljamo o zmožnostih postopka *kvadratne metode* M_{split} v 2D deformacijski analizi. V nasprotju s prejšnjimi raziskavami, ki so se osredotočale na nivelmanske mreže, se naša študija osredotoča na praktično uporabo postopka *kvadratne metode* M_{split} v kotno-dolžinskih horizontalnih geodetskih mrežah, podprto s temeljitimi teoretičnimi in empiričnimi analizami.

Po izpeljavi in prikazu enačb, na katerih temelji postopek *kvadratne metode* M_{split}, smo izvedli izračun na isti simulirani 2D geodetski mreži kot v deformacijski analizi po postopkih Hannover, Karlsruhe, Delft, Fredericton, München in Caspary.

V postopku *kvadratne metode* M_{split} smo najprej izbrali začetne vrednosti $\mathbf{x}_\alpha^0$ iz izravnave meritev po metodi najmanjših kvadratov $\hat{\mathbf{x}}_{MNK}$, torej $\mathbf{x}_\alpha^0 = \hat{\mathbf{x}}_{MNK}$ in $\mathbf{v}_\alpha^0 = \hat{\mathbf{v}}_{MNK}$. Izkazalo se je, da bi lahko izbrali tudi

20-krat večje vrednosti ($\mathbf{x}_\alpha^0 = 20 \cdot \hat{\mathbf{x}}_{MNK}$) ali 1000-krat manjše vrednosti ($\mathbf{x}_\alpha^0 = \hat{\mathbf{x}}_{MNK}/1000$) in bi vedno dobili enak končni rezultat (le število iteracij za izračun končne rešitve bi bilo drugačno). Nato smo izračunali $\mathbf{x}_\beta^0$ in $\mathbf{v}_\beta^0$ po enačbah (58) in (59). Iteracijski postopek izračuna končnih vrednosti popravkov približnih vrednosti koordinatnih neznank smo izvedli po enačbah od (46) do (53) ter dobili končni rezultat v 8. iteracijah. V tem koraku so se vrednosti $\mathbf{x}_\alpha^8$ in $\mathbf{x}_\beta^8$ razlikovale od $\mathbf{x}_\alpha^7$ in $\mathbf{x}_\beta^7$ za manj od izbrane meje za končanje iteracijskega postopka $\varepsilon = 0{,}001$. Na koncu smo izračunali premike posameznih točk v obravnavani geodetski mreži po enačbi (63) in rezultate primerjali z rezultati deformacijskih analiz po postopkih Hannover, Karlsruhe, Delft, Fredericton, München, Caspary in s simuliranimi rezultati.

Iz preglednice 4 vidimo, da so premiki točk primerljivi s simuliranimi. Primerljivost rezultatov vidimo tudi na sliki 3. Na podlagi izračunanih rezultatov lahko ugotovimo, da je deformacijska analiza po postopku *kvadratne metode* M_{split} uporabna za določitev premikov točk v 2D geodetski mreži.

V primerjavi z drugimi naštetimi postopki ima *kvadratna metoda* M_{split} pomanjkljivost, da nima testne statistike, na podlagi katere bi lahko ugotovili, ali gre za statistično značilen premik ali ne. Prednosti *kvadratne metode* M_{split} pa so, da ne zahteva vnaprejšnjih predpostavk glede točk mreže, ali so pri miru ali se premikajo. Vsekakor pa je *kvadratna metoda* M_{split} dodatna uporabna metoda deformacijske analize, kar pride prav pri težavnih primerih, ko so premiki točk v 2D mrežah majhni.

V prihodnosti bomo preverili delovanje metode pri več kot dveh terminskih izmerah. Prav tako nameravamo metodo preizkusiti na 3D geodetski mreži, kjer se hkrati obravnavajo tudi podatki o višinah točk.

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